Spectra of Digital Waveform

time-domain and frequency-domain

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Basic Properties of Fourier Transform

1. periodic $x(t)$

$$F(\, x(t)\,) \;=\; 1/T\,\int_0^T x(t)e^{-jn\omega_0 t}dt \;=\; C_n$$

$$\therefore\; x(t) \;=\; \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

FIGURE 7.1 A periodic signal with period T.

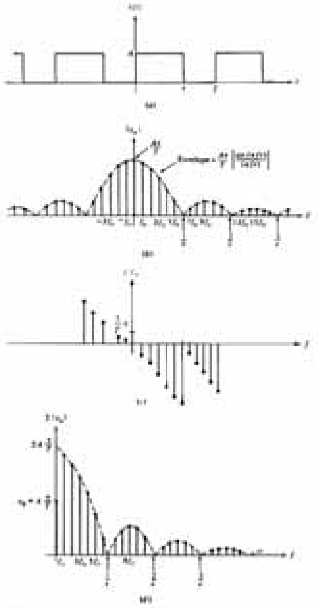


FIGURE 7.6 Frequency-domain representation of a square wave: (a) the signal; (b) the two-sided magnitude spectrum; (c) the phase spectrum; (d) the one-sided magnitude spectrum [1].

Basic Properties of Fourier Transform

2. $C_n = C_{-n}^*$ for real $x(t)$

3. $F\,(x_1(t) + x_2(t)) = F\,(x_1(t)) + F\,(x_2(t))$

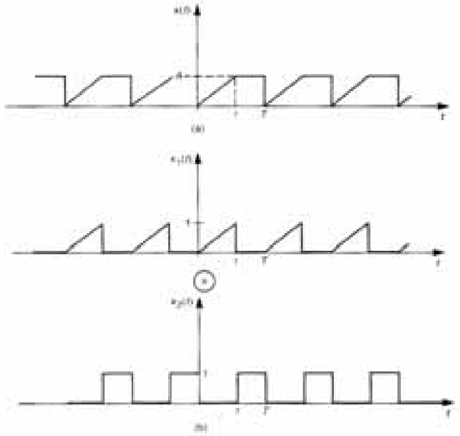


FIGURE 7.9 Illustration of the principle of linear decomposition of a signal: (a) the signal; (b) its decomposition, $x(t) = Ax_1(t) + Ax_2(t)$.

Basic Properties of Fourier Transform

4. $F\ (\ x(t-\alpha)\ )\ =\ e^{-jn\omega_0\alpha}\cdot F\ (\ x(t)\ )$

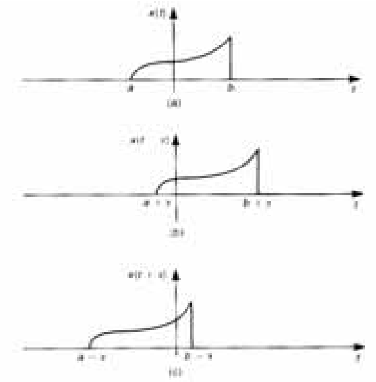


FIGURE 7.10 Illustration of the time-shift principle: (a) x(t); (b) x(t − α); (c) x(t + α).

Basic Properties of Fourier Transform

5. $F\ (\ \delta(\mathrm{t}\pm kT)\ )\ \Rightarrow\ \mathrm{C_n}=(1/T)e^{-j\omega_0\alpha}$ $k=0,\pm1,\pm2,\pm3....$

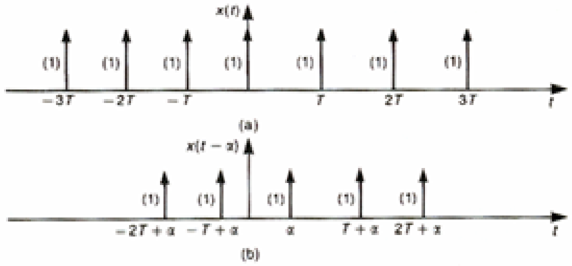
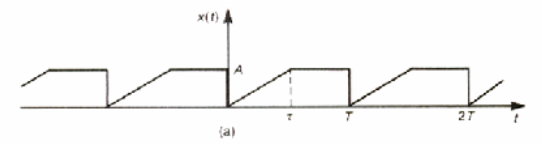


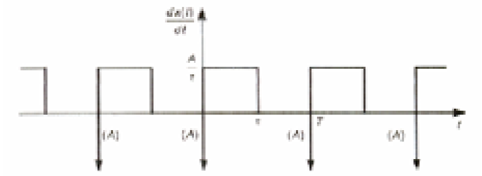
FIGURE 7.12 A periodic train of unit impulses (a) and (b) shifted in time.

Basic Properties of Fourier Transform

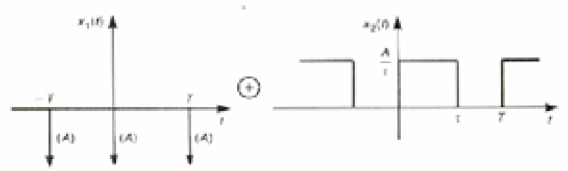
6. $F\ (\ d^k x(t)/dt^k\ )\ \Rightarrow\ C_n^{(k)} = (\ jn\omega_0)^k C_n$

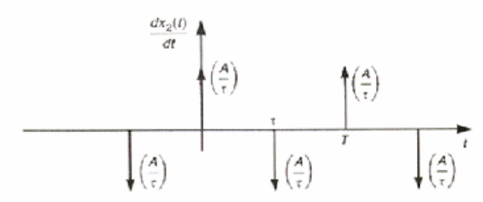


(a)



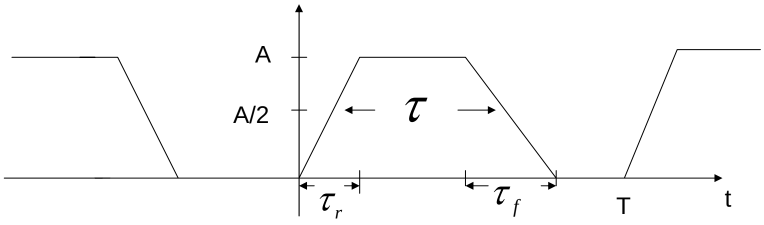
Basic Properties of Fourier Transform





Spectra of digital circuit waveforms

a. The periodic , trapezoidal pulse train representing clock and data signals of digital systems.



A: Amplitude, τ_r : pulse rise time , τ_f : pulse fall time

τ : pulse width (50 %)

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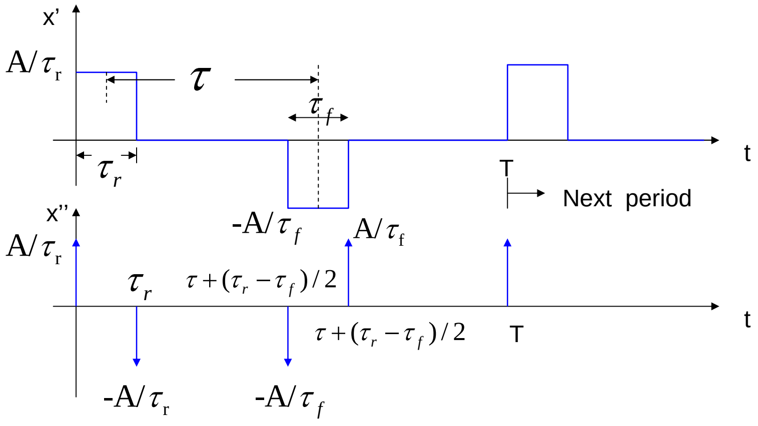
c. To obtain the complex exponential Fourier series of this waveform

(1)



(2)

$$\therefore \quad F(x'') = A/T\tau_r - A e^{-jn\omega_0\tau_r}/T\tau_r - A e^{-jn\omega_0(\tau+(\tau_r-\tau_f)/2)}/T\tau_f$$

$$+ A e^{-jn\omega_0(\tau+(\tau_r+\tau_f)/2)}/T\tau_f$$

$$= A/T * [\,(e^{-jn\omega_0\tau_r/2}/\tau_r)*(e^{jn\omega_0\tau_r/2} - e^{-jn\omega_0\tau_r/2})$$

$$- (e^{-jn\omega_0\tau_r/2}e^{-jn\omega_0\tau}/\tau_f) * (e^{jn\omega_0\tau_f/2} - e^{-jn\omega_0\tau_f/2})\,]$$

$$= j\,(A/2\pi n)*(n\omega_0)^2 * e^{-jn\omega_0(\tau_r+\tau_f)/2} * [\sin(n\omega_0\tau_r/2)*e^{jn\omega_0\tau/2}/(n\omega_0\tau_r/2$$

$$- \sin(n\omega_0\tau_f/2)*e^{-jn\omega_0\tau/2}/(n\omega_0\tau_f/2)\,]$$

(3)

$$F(x) = F(x'')/(jn\omega_0)^2 \quad , \quad n \neq 0$$

$$if \quad \tau_r = \tau_f$$

$$C_n = A\frac{\tau}{T}\,\frac{\sin(n\omega_0\tau/2)}{n\omega_0\tau/2}\,\frac{\sin(n\omega_0\tau_r/2)}{(n\omega_0\tau_r/2)}\,e^{-jn\omega_0(\tau+\tau_r)/2}$$

---

d. Expression as one-side spectrum

$$x(t) = C_0 + \sum_{n=1}^{\infty}\left|C_n^{\,+}\right|\cos(n\omega_0 t + \angle C_n)$$

when $\left|C_n^{\,+}\right| = 2\left|C_n\right|$

$$= 2A\tau\left|\sin(n\pi\tau/T)/(n\pi\tau/T)\right|\left|\sin(n\pi\tau_r/T)/(n\pi\tau_r/T)\right| \quad , \ for \ \ n \neq 0$$

with $\tau_r = \tau_f$, $C_0 = A\tau/T$, and $\omega_0 = 2\pi/T$

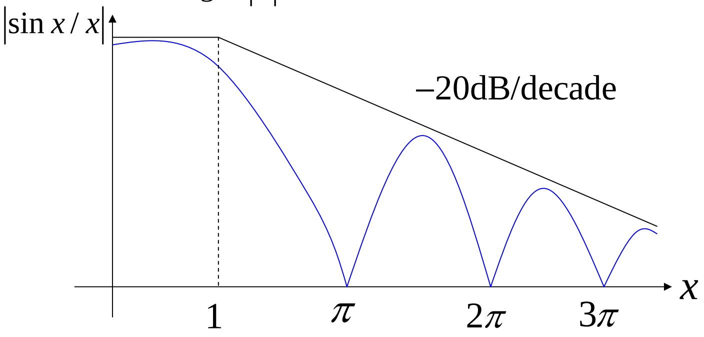
Note : (1) 50% duty cycle $\Leftrightarrow$ $\tau/T = 1/2$ $\Leftrightarrow$ no even harmonics

(2) slight variations from an 50% $\Rightarrow$ even harmonics vary widely

Spectral bounds for trapezoidal waveforms

$$\text{a.}\quad |\sin x/x| \le \left\langle \begin{array}{ll} 1 \ , & \text{for small } x \\ 1/|x| \ , & \text{for small } x \end{array} \right.$$

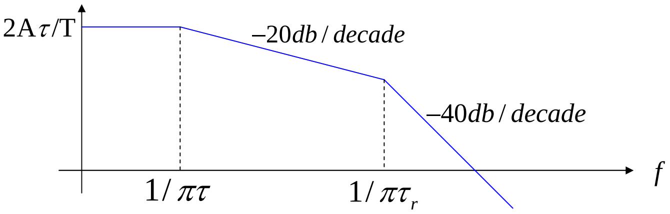
$$\simeq 20\log 1/|x| = -20\text{dB/decade} \ , \quad \text{for small } x$$



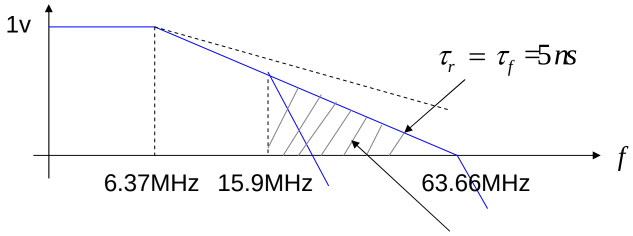
Effect of rise/fall time on Spectral Content

b. let $f = n/T$ $\therefore$ envolope = $2A{*}\tau|\sin(\pi\tau f)/(\pi\tau f)||\sin(\pi\tau_r f)/(\pi\tau_r f)| / T$

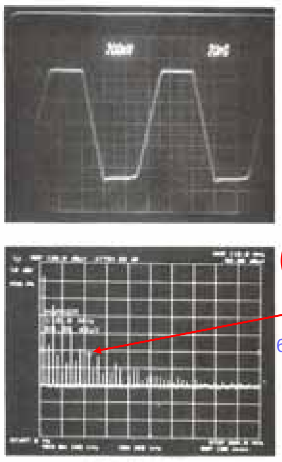
$\Rightarrow$ 20log(envolope) = $20\log(2A\tau / T) + 20\log |\sin(\pi\tau f)/(\pi\tau f)| + 20\log|\sin(\pi\tau_r f)/(\pi\tau_r f)|$



Effect of rise/fall time on Spectral Content



Effect of rise/fall time on Spectral Content



Rise/fall time = 20 ns
(a)

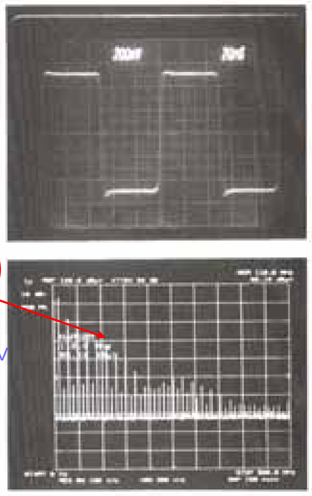
FIGURE 7.19 Experimentally measured spectra of 1 V, 10 MHz, 50% duty cycle trapezoidal pulse trains for rise/falltimes of (a) 20 ns.

20ns

11th harmonic

110MHz

68dBuV 86.1dBuV



Rise/fall time = 5 ns
(b)

FIGURE 7.19 continued. (b) 5 ns.

5ns

Effect of rise/fall time on Spectral Content

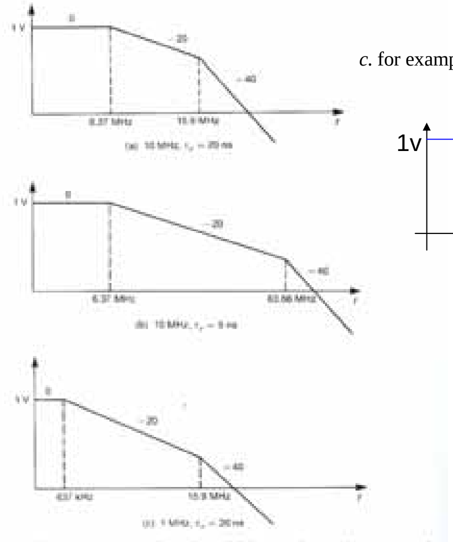
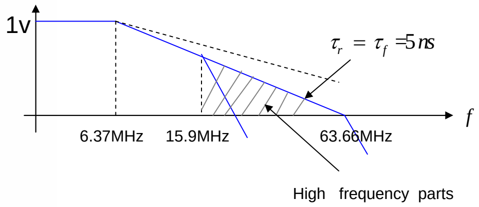


FIGURE 7.20 Illustration of the effect of rise/falltimes and repetition rate on the spectral content of 1 V, 50% duty cycle trapezoidal pulse trains: (a) 10 MHz, $\tau_r = \tau_f = 20$ ns; (b) 10 MHz, $\tau_r = \tau_f = 5$ ns; (c) 1 MHz, $\tau_r = \tau_f = 20$ ns.

c. for example : 10MHz , $\tau_r \ = \ \tau_f$ =20 *ns* *and* $\tau_r \ = \ \tau_f$ =5 *ns*



Effect of rise/fall time on Spectral Content

Spectral bound prediction

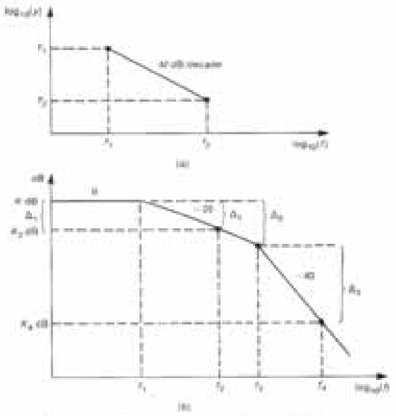


FIGURE 7.21 Illustration of interpolation on log–log (Bode) plots: (a) the general characterization of a linear segment; (b) application to the spectral bound for a trapezoidal pulse train.

20ns rise time

$$\mathrm{Level}_{110\mathrm{MHz}} = 20\log_{10}(1V) - 20\log_{10}(\frac{15.9MHz}{6.37MHz}) - 40(\frac{110MHz}{15.9MHz})$$
$$= 78.45dB\mu V$$

5ns rise time

$$\mathrm{Level}_{110\mathrm{MHz}} = 20\log_{10}(1V) - 20\log_{10}(\frac{63.7MHz}{63.7MHz}) - 40(\frac{110MHz}{63.7MHz})$$
$$= 90.5dB\mu V$$

Effect of rise/fall time on Spectral Content

The exact value calculated by *sinc* function

20ns	73.8dBuV
5ns	90.4dBuV

Very close to the spectral bound prediction

The measured values are

20ns	68.8dBuV
5ns	86.1dBuV

Why are they lower than the value by the exact calculations?

What is the effect of the duty cycle on the spectral bound?

Effect of Repetition rate and Duty Cycle

What is the effect of the duty cycle on the spectral bound?

Duty cycle:

$$D = \frac{\tau}{T}$$

One-sided spectrum:

$$\left|c_n^+\right| = 2AD\left|\frac{\sin(n\pi D)}{n\pi D}\right|\left|\frac{n\pi\tau_r f_0}{n\pi\tau_r f_0}\right|$$

- DC level
- 1st breakpoint

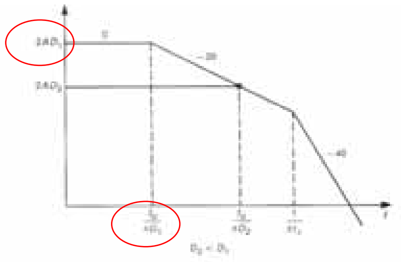


FIGURE 7.22 Illustration of the effect of duty cycle on the spectral bounds of trapezoidal pulse trains.

Effect of Ringing (Undershoot/Overshoot)

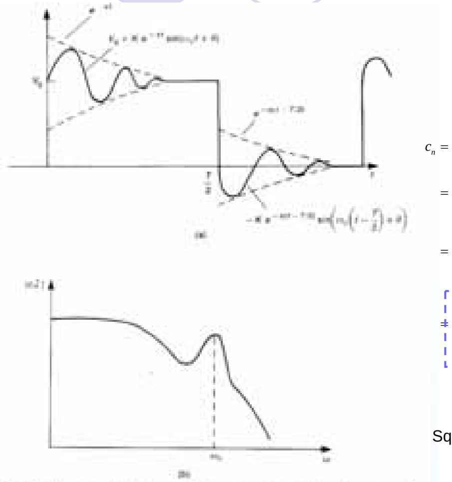


FIGURE 7.23 Illustration of ringing (undershoot/overshoot): (a) time-domain waveform; (b) spectral content.

Assume θ = 0

$$c_n = c_{n(squarewave)} + \frac{1}{T}\int_0^{T/2} Ke^{-\alpha t}\sin(\omega_r t+\theta)e^{-jn\omega_0 t}dt - e^{-jn\omega_0 T/2}\left[\frac{1}{T}\int_0^{T/2} Ke^{-\alpha t}\sin(\omega_r t+\theta)e^{-jn\omega_0 t}dt\right]$$

$$= c_{n(squarewave)} + (1-e^{-jn\omega_0 T/2})\frac{1}{T}\int_0^{T/2} Ke^{-\alpha t}\sin(\omega_r t+\theta)e^{-jn\omega_0 t}dt$$

$$= \frac{V_0}{2}\frac{\sin(\frac{1}{4}n\omega_0 T)}{\frac{1}{4}n\omega_0 T}e^{-jn\omega_0 T/4} + \frac{K}{2}\frac{\sin(\frac{1}{4}n\omega_0 T)}{\frac{1}{4}n\omega_0 T}e^{-jn\omega_0 T/4}\frac{p\omega_r}{p^2+2\alpha p+\alpha^2+\omega_r^2}$$

$$= \underbrace{\boxed{\frac{V_0}{2}\frac{\sin(\frac{1}{4}n\omega_0 T)}{\frac{1}{4}n\omega_0 T}}e^{-jn\omega_0 T/4}}_{\text{Square wave components}}\underbrace{\frac{p^2+(2\alpha+(K/V_0)\omega_r)p+(\alpha^2+\omega_r^2)}{p^2+2\alpha p+(\alpha^2+\omega_r^2)}}_{\text{Bandpass}}$$

where $p = jn\omega_0$

Effect of Ringing (Undershoot/Overshoot)

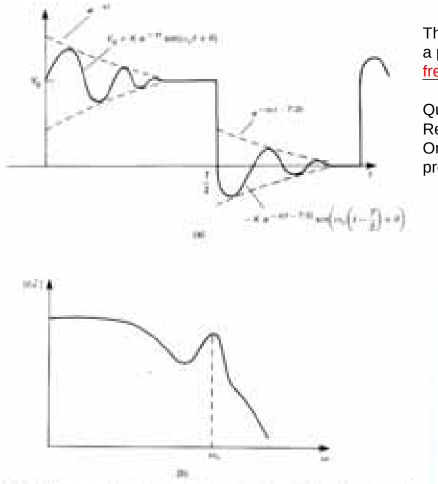


FIGURE 7.23 Illustration of ringing (undershoot/overshoot): (a) time-domain waveform; (b) spectral content.

The spectrum of a square wave with undershoot/overshoot has a part of its spectrum enhanced or increased about the ringing frequency W_r.

Quite often we see in the radiated emission profile a seemingly Resonant region of enhanced emissions in a narrow frequency band. One possible explanation for this is the undershoot/overshoot present on the digital waveforms.

Use of Spectral Bounds in Computing Bounds on the Output Spectrum of Linear System

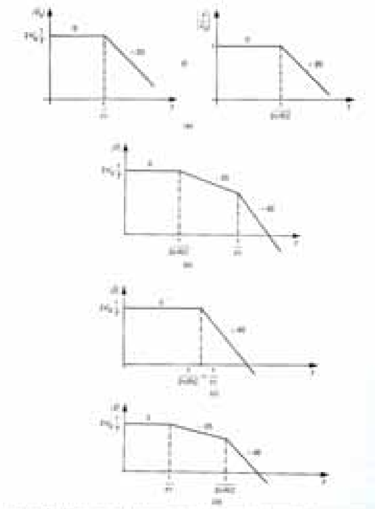


FIGURE 7.24 Illustration of the estimation of processing of a signal by linear systems using spectral bounds and Bode plots for a square-wave input and a low-pass RC circuit: (a) the input and the transfer function; (b) RC > ½τ; (c) RC = ½τ; (d) RC < ½τ.

Spectrum Analyzer

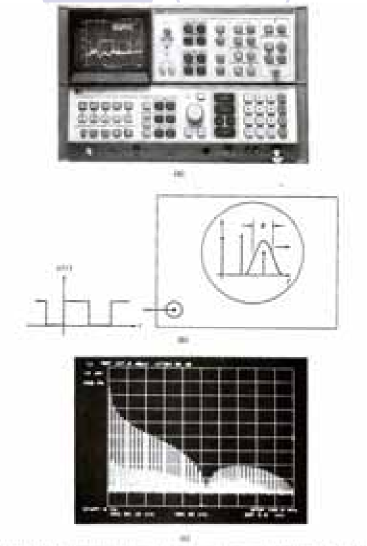
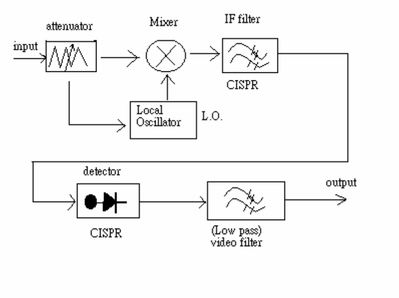


FIGURE 7.23 A spectrum analyzer: (a) photograph (courtesy of the Hewlett Packard Company); (b) illustration of the function as a swept bandpass filter; (c) measured spectrum of a 1 V, 1 MHz, 50% duty cycle, $t_r = t_f = 12.5$ ns, trapezoidal pulse train [1].



Spectrum Analyzer

Bandwidth of SA

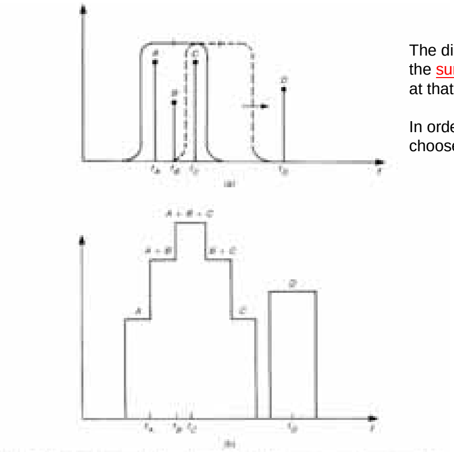


FIGURE 7.26 Illustration of the effect of bandwidth on measured spectrum: (a) a bandwidth that is wide enough to contain several narrowband signals; (b) the resulting display.

The displayed level at the center frequency of the bandwidth will be the sum of the spectral levels that fall within the bandwidth of the filter at that time.

In order to obtain the lowest possible level on the SA display, we should choose as small a bandwidth as possible.

Spectrum Analyzer

EMC receiver regulation

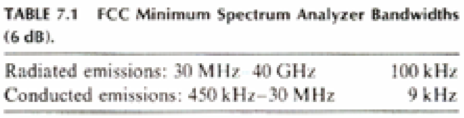


TABLE 7.1 **FCC Minimum Spectrum Analyzer Bandwidths (6 dB).**

Radiated emissions: 30 MHz–40 GHz	100 kHz
Conducted emissions: 450 kHz–30 MHz	9 kHz

TABLE 7.2 **CISPR 22 Minimum Spectrum Analyzer Bandwidths (6 dB).**

Radiated emissions: 30 MHz–1 GHz	100 kHz
Conducted emissions: 150 kHz–30 MHz	9 kHz

Spectrum Analyzer

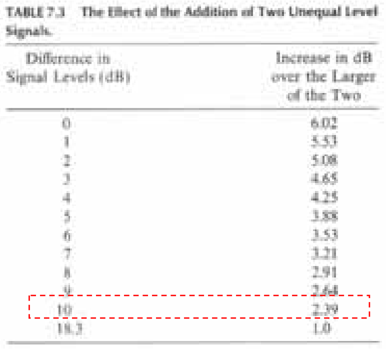


TABLE 7.3 The Effect of the Addition of Two Unequal Level Signals.

Difference in Signal Levels (dB)	Increase in dB over the Larger of the Two
0	6.02
1	5.53
2	5.08
3	4.65
4	4.25
5	3.88
6	3.53
7	3.21
8	2.91
9	2.64
10	2.39
18.3	1.0

Spectrum Analyzer

Peak and Quasi-peak

Infrequently occurring signals will result in a measured Quasi-peak level that is considerably smaller than a peak detector would give.

That infrequent events may be of sufficient magnitude distressingly large received levels on a SA that is set to peak detector function.

The reason for the use of quasi-peak detector function relates to the intent of the regulatory limits, which is to prevent interference in radio and wire communication Receivers. Infrequent spikes and other events do not substantially prevent the listener from obtaining the desired information.

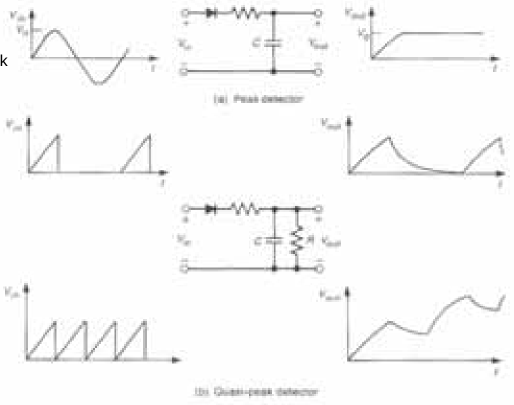


FIGURE 7.27 Two important detectors: (a) the peak detector; (b) the quasi-peak detector.

Representation of Random Signals

Digital data waveforms are obviously random signals, not deterministic signals

$$x(t) = \frac{1}{2} X_0 [1 + m(t)]$$

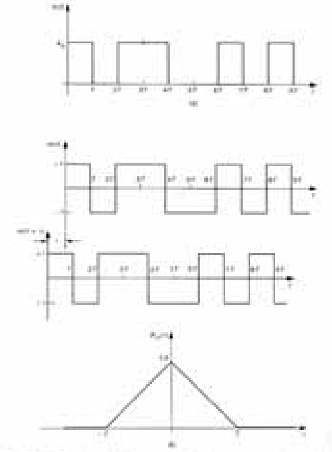


FIGURE 7.30 Illustration of the computation of the power spectral density of a PCM–NRZ signal: (a) a typical waveform, (b) the autocorrelation function.

$$\begin{aligned} R_x(\tau) &= \overline{x(t)x(t+\tau)} \\ &= \lim_{t\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)dt \\ &= \frac{1}{4} X_0^2 \overline{[1+m(t)][1+m(t+\tau)]} \\ &= \frac{1}{4} X_0^2 [1 + \overline{m(t)} + \overline{m(t+\tau)} + \overline{m(t)m(t+\tau)}] \\ &= \frac{1}{4} X_0^2 [1 + \overline{m(t)m(t+\tau)}] \\ &= 1 - \frac{|\tau|}{T} \qquad \text{for } |\tau| < T \\ &= 0 \qquad \text{for } |\tau| > T \end{aligned}$$

stationary

Representation of Random Signals

Power Spectral density

$$G_x(f) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j\omega\tau} d\tau$$

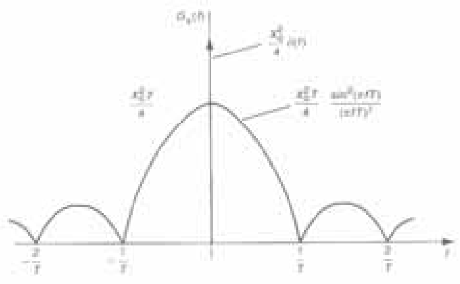


FIGURE 7.31 The power spectral density of the PCM–NRZ signal.