Microwave Filter Design

Chp 8. Coupled Resonator Circuits

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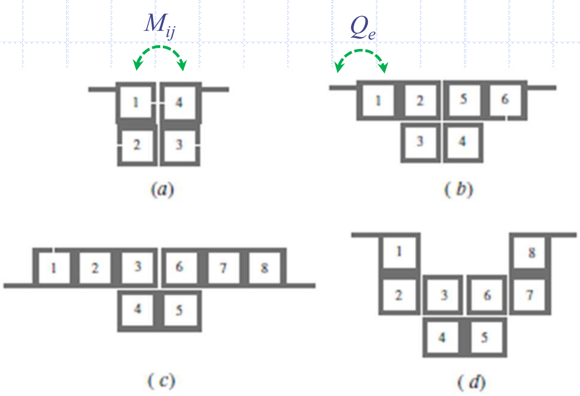
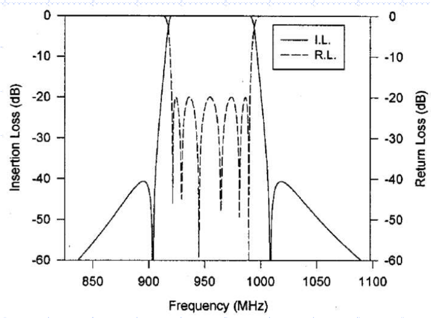
Coupled Resonator Circuits

- Features
  Suitable to design narrow-band filters
- Design parameters
  1. Coupling matrix M
     relationship between coupling coefficients and physical structure
  2. External quality factor Q_e
     the externally loaded input/output resonators





General Coupling Matrix for Coupled-Resonators (1/4)

➤ Loop Equation Formulation (magnetic coupling)

KVL:

$$\left(R_1 + j\omega L_1 + \frac{1}{j\omega C_1}\right) i_1 - j\omega L_{12} i_2 \cdots - j\omega L_{1n} i_n = e_s$$

$$-j\omega L_{21} i_1 + \left(j\omega L_2 + \frac{1}{j\omega C_2}\right) i_2 \cdots - j\omega L_{2n} i_n = 0$$

$$\vdots$$

$$-j\omega L_{n1} i_1 - j\omega L_{n2} i_2 \cdots + \left(R_n + j\omega L_n + \frac{1}{j\omega C_n}\right) i_n = 0$$

$$\Rightarrow \begin{bmatrix} R_1 + j\omega L_1 + \frac{1}{j\omega C_1} & -j\omega L_{12} & \cdots & -j\omega L_{1n} \\ -j\omega L_{21} & j\omega L_2 + \frac{1}{j\omega C_2} & \cdots & -j\omega L_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -j\omega L_{n1} & -j\omega L_{n2} & \cdots & R_n + j\omega L_n + \frac{1}{j\omega C_n} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{bmatrix} = \begin{bmatrix} e_s \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or

$$[Z]\cdot[i] = [e]$$

General Coupling Matrix for Coupled-Resonators (2/4)

- Synchronously tuned filter

  All the resonators resonates at the same frequency

$$\omega_0 = 1/\sqrt{LC} \Rightarrow L = L_1 = L_2 = \cdots L_n$$

and

$$C = C_1 = C_2 = \cdots C_n$$

- Simplified [Z]

$$[Z] = \omega_0 L \cdot FBW \cdot [\bar{Z}]$$

$$\begin{bmatrix} R_1 + j\omega L_1 + \frac{1}{j\omega C_1} & -j\omega L_{12} & \cdots & -j\omega L_{1n} \\ -j\omega L_{21} & j\omega L_2 + \frac{1}{j\omega C_2} & \cdots & -j\omega L_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -j\omega L_{n1} & -j\omega L_{n2} & \cdots & R_n + j\omega L_n + \frac{1}{j\omega C_n} \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{R_1}{\omega_0 L \cdot FBW} + p & -j\frac{\omega}{\omega_0}\frac{L_{12}}{L}\cdot\frac{1}{FBW} & \cdots & -j\frac{\omega}{\omega_0}\frac{L_{1n}}{L}\cdot\frac{1}{FBW} \\ -j\frac{\omega}{\omega_0}\frac{L_{21}}{L}\cdot\frac{1}{FBW} & p & \cdots & -j\frac{\omega}{\omega_0}\frac{L_{2n}}{L}\cdot\frac{1}{FBW} \\ \vdots & \vdots & \vdots & \vdots \\ -j\frac{\omega}{\omega_0}\frac{L_{n1}}{L}\cdot\frac{1}{FBW} & -j\frac{\omega}{\omega_0}\frac{L_{n2}}{L}\cdot\frac{1}{FBW} & \cdots & \frac{R_n}{\omega_0 L \cdot FBW} + p \end{bmatrix}$$

$$j\omega L + \frac{1}{j\omega C} = j\omega L + \frac{\omega_0^2 L}{j\omega} = j\omega_0 L\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

with

$$p = j\frac{1}{FBW}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

Bandpass transformation!

General Coupling Matrix for Coupled-Resonators (3/4)

$$[\bar{Z}] = \begin{bmatrix} \frac{R_1}{\omega_0 L \cdot FBW} + p & -j\frac{\omega}{\omega_0}\frac{L_{12}}{L}\cdot\frac{1}{FBW} & \cdots & -j\frac{\omega}{\omega_0}\frac{L_{1n}}{L}\cdot\frac{1}{FBW} \\ -j\frac{\omega}{\omega_0}\frac{L_{21}}{L}\cdot\frac{1}{FBW} & p & \cdots & -j\frac{\omega}{\omega_0}\frac{L_{2n}}{L}\cdot\frac{1}{FBW} \\ \vdots & \vdots & \vdots & \vdots \\ -j\frac{\omega}{\omega_0}\frac{L_{n1}}{L}\cdot\frac{1}{FBW} & -j\frac{\omega}{\omega_0}\frac{L_{n2}}{L}\cdot\frac{1}{FBW} & \cdots & \frac{R_n}{\omega_0 L \cdot FBW} + p \end{bmatrix} \Rightarrow [\bar{Z}] = \begin{bmatrix} \frac{1}{q_{e1}} + p & -jm_{12} & \cdots & -jm_{1n} \\ -jm_{21} & p & \cdots & -jm_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -jm_{n1} & -jm_{n2} & \cdots & \frac{1}{q_{en}} + p \end{bmatrix}$$

- External quality factor

$$\frac{R_i}{\omega_0 L} = \frac{1}{Q_{ei}} \qquad \text{for } i = 1, n$$

- Scaled external quality factor

$$q_{ei} = Q_{ei} \cdot FBW \qquad \text{for } i = 1, n$$

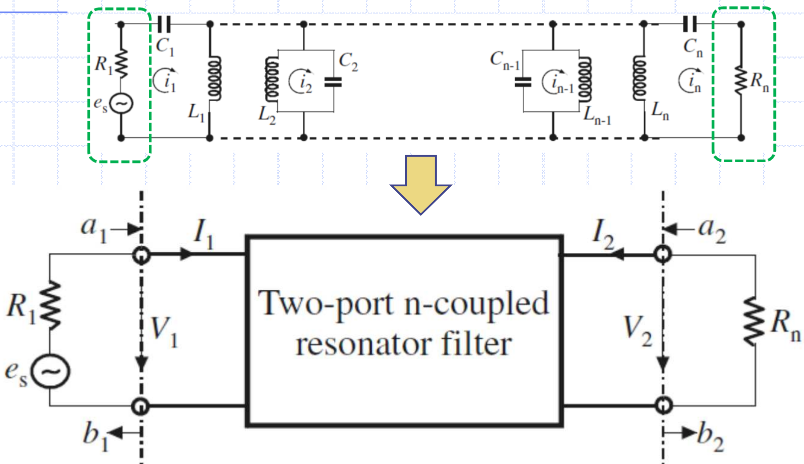
- Coupling coefficient

$$M_{ij} = \frac{L_{ij}}{L}$$

- Normalized coupling coefficient

$$m_{ij} = \frac{M_{ij}}{FBW}$$

General Coupling Matrix for Coupled-Resonators (4/4)



$$V_1 = e_s - i_1 R_1 \qquad I_1 = i_1$$

$$V_2 = i_n R_n \qquad I_2 = -i_n$$

$$\Rightarrow \quad a_1 = \frac{e_s}{2\sqrt{R_1}} \qquad b_1 = \frac{e_s - 2i_1 R_1}{2\sqrt{R_1}}$$

$$a_2 = 0 \qquad b_2 = i_n\sqrt{R_n}$$

$$a_n = \frac{1}{2}\left(\frac{V_n}{\sqrt{Z_{0n}}} + \sqrt{Z_{0n}}\, I_n\right)$$

$$b_n = \frac{1}{2}\left(\frac{V_n}{\sqrt{Z_{0n}}} - \sqrt{Z_{0n}}\, I_n\right)$$

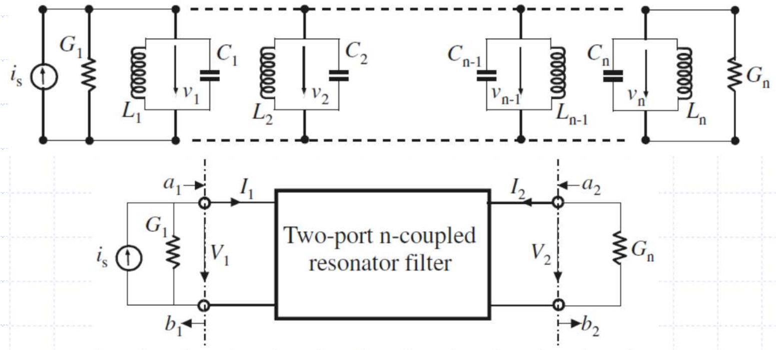
$$\Rightarrow \quad S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} = \frac{2\sqrt{R_1 R_n}\, i_n}{e_s} = \frac{2\sqrt{R_1 R_n}}{\omega_0 L \cdot FBW}[\bar{Z}]_{n1}^{-1} = 2\frac{1}{\sqrt{q_{e1}\cdot q_{en}}}[\bar{Z}]_{n1}^{-1}$$

$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} = 1 - \frac{2R_1 i_1}{e_s} = 1 - \frac{2R_1}{\omega_0 L \cdot FBW}[\bar{Z}]_{11}^{-1} = 1 - \frac{2}{q_{e1}}[\bar{Z}]_{11}^{-1}$$

Solve the matrix: $\omega_0 L \cdot FBW \cdot [\bar{Z}] \cdot [i] = [e]$

General Coupling Matrix

- Similar results for node equation formulation (electric coupling) can be obtained



$$S_{21} = 2\frac{1}{\sqrt{q_{e1} \cdot q_{en}}}[\bar{Y}]_{n1}^{-1}$$

$$S_{11} = \frac{2}{q_{e1}}[\bar{Y}]_{11}^{-1} - 1$$

- Both magnetic or electric can be incorporated into one form

$$S_{21} = 2\frac{1}{\sqrt{q_{e1} \cdot q_{en}}}[A]_{n1}^{-1}$$

$$S_{11} = \pm\left(1 - \frac{2}{q_{e1}}[A]_{11}^{-1}\right)$$

with $[A] = \underbrace{[q]}_{\text{external quality factor}} + p[U] - j\underbrace{[m]}_{\text{general coupling matrix}}$

- External quality factor and general coupling matrix can be obtained from the required frequency response

Comments

For a given filtering characteristic of *S21(p) and S11(p), the coupling matrix and* the external quality factors may be obtained using the synthesis procedure developed in [10–11].

However, the elements of the coupling matrix *[m] that emerge* from the synthesis procedure will, in general, all have nonzero values.

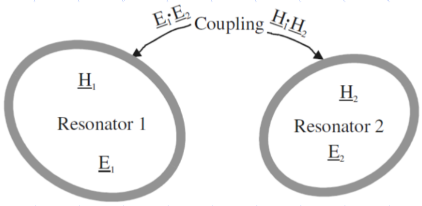
But, a nonzero entry everywhere else means that in the network that *[m] represents, couplings exist between every resonator and every other* resonator.

As this is clearly impractical, it is usually necessary to perform a sequence of similar transformations until a more convenient form for implementation is obtained.

A more practical synthesis approach based on optimization will be presented in the next chapter.

General Theory of Couplings

- Coupling coefficients of coupled resonators (establish the relationship between the value of required coupling and the physical structure)



electric coupling | magnetic coupling

$$k = \frac{\iiint \varepsilon \underline{E}_1 \cdot \underline{E}_2 dv}{\sqrt{\iiint \varepsilon |\underline{E}_1|^2 dv \times \iiint \varepsilon |\underline{E}_2|^2 dv}} + \frac{\iiint \mu \underline{H}_1 \cdot \underline{H}_2 dv}{\sqrt{\iiint \mu |\underline{H}_1|^2 dv \times \iiint \mu |\underline{H}_2|^2 dv}}$$

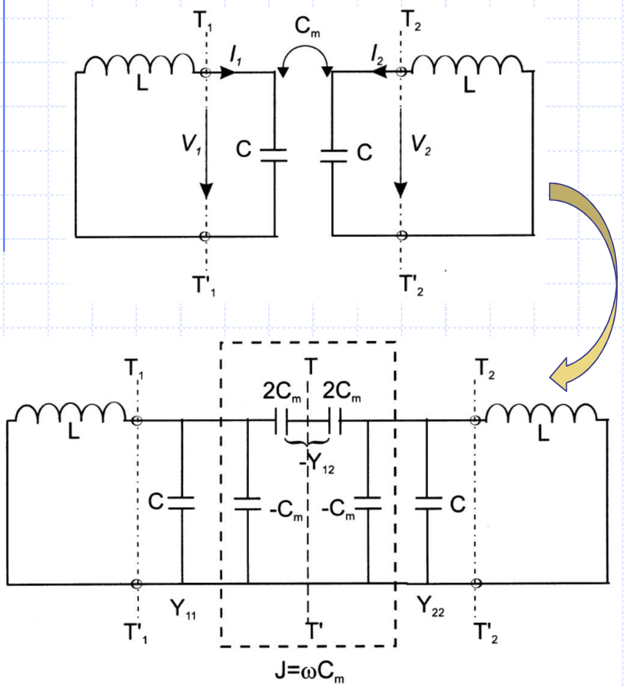
- Positive sign implies that coupling enhances the stored energy of uncoupled resonator.
- Negative sign indicates that coupling reduceds the stored energy of uncoupled resonator.

- In fact, this is not an easy task to evaluate the coupling coefficients, since it requires knowledge of the field distributions and the performance of the space integral.

- Simplified lumped-element circuit model can be used to facilitate the analysis of coupling coefficients on a narrow-band basis.

Synchronously Tuned Coupled-Resonator Circuit - Electric Coupling

- Electric coupling coefficients of coupled resonators



at the reference plane T_1-T_1' and T_2-T_2'

$$I_1 = j\omega C V_1 - j\omega C_m V_2$$
$$I_2 = j\omega C V_2 - j\omega C_m V_1$$

$$\Downarrow$$

$$Y_{11} = Y_{22} = j\omega C$$
$$Y_{12} = Y_{21} = -j\omega C_m$$

at the reference plane T-T'

Electric wall:

$$f_e = \frac{1}{2\pi\sqrt{L(C+C_m)}}$$

Magnetic wall:

$$f_m = \frac{1}{2\pi\sqrt{L(C-C_m)}}$$

Electrical coupling coefficients can be obtained:

$$k_E = \frac{f_m^2 - f_e^2}{f_m^2 + f_e^2} = \frac{C_m}{C}$$

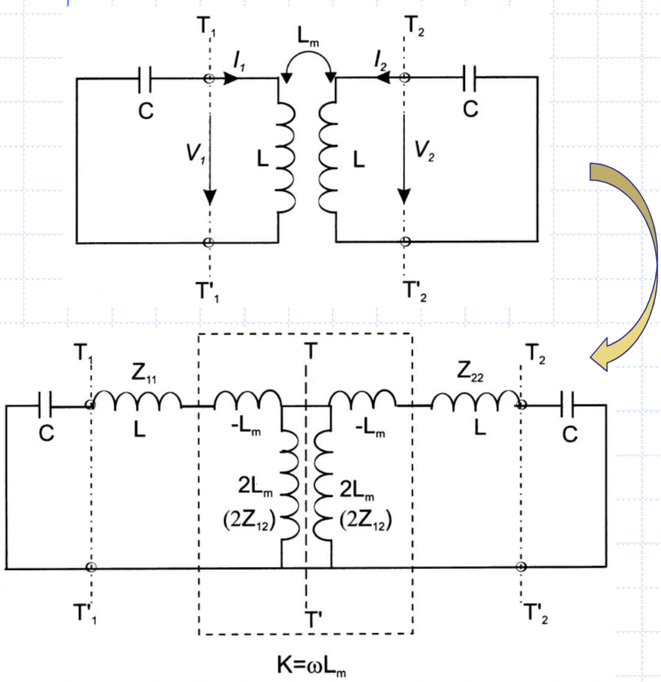
Comment

It might be well to mention that (8.32) implies that <u>the self-capacitance C is the capacitance seen in one resonant loop of Figure 8.4(*a*) when the capacitance in the adjacent loop is shorted out</u>.

Thus, the second terms on the R.H.S. of (8.32) are the induced currents resulting from the increasing voltage in resonant loop 2 and loop 1, respectively.

Synchronously Tuned Coupled-Resonator Circuit - Magnetic Coupling

- Magnetic coupling coefficients of coupled resonators



at the reference plane T_1-T_1' and T_2-T_2'

$$V_1 = j\omega L I_1 + j\omega L_m I_2$$

$$V_2 = j\omega L I_2 + j\omega L_m I_1$$

$$Z_{11} = Z_{22} = j\omega L$$

$$Z_{12} = Z_{21} = j\omega L_m$$

at the reference plane T-T'

Electric wall:

$$f_e = \frac{1}{2\pi\sqrt{(L - L_m)C}}$$

Magnetic wall:

$$f_m = \frac{1}{2\pi\sqrt{(L + L_m)C}}$$

Electrical coupling coefficients can be obtained:

$$k_M = \frac{f_e^2 - f_m^2}{f_e^2 + f_m^2} = \frac{L_m}{L}$$

Comment

The equations in (8.37) also imply that the self-inductance *L is the inductance seen* in one resonant loop of Figure 8.5(*a*) *when the adjacent loop is open-circuited.*

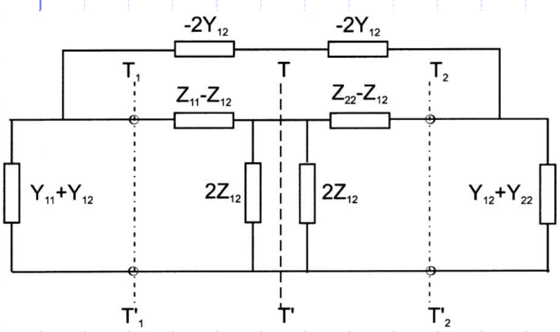
Thus, the second terms on the R.H.S. of (8.37) are the induced voltages resulting from the increasing current in loops 2 and 1, respectively.

It should be noticed that the two loop currents in Figure 8.5(*a*) *flow in the opposite directions, so that the* voltage drops due to the mutual inductance have a positive sign.

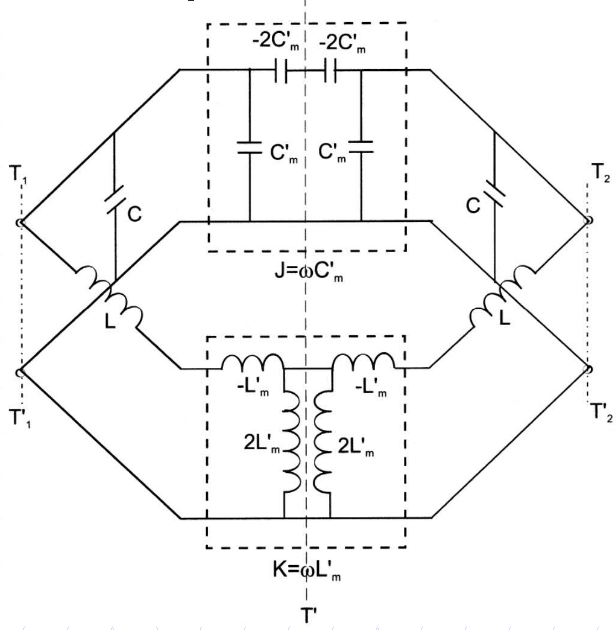
Synchronously Tuned Coupled-Resonator Circuit - Mixed Coupling (1/2)

- Mixed coupling coefficients of coupled resonators

  Both the electric and magnetic couplings cannot be ignored



Equivalent circuit model



Electric coupling:

$$Y_{11} = Y_{22} = j\omega C$$

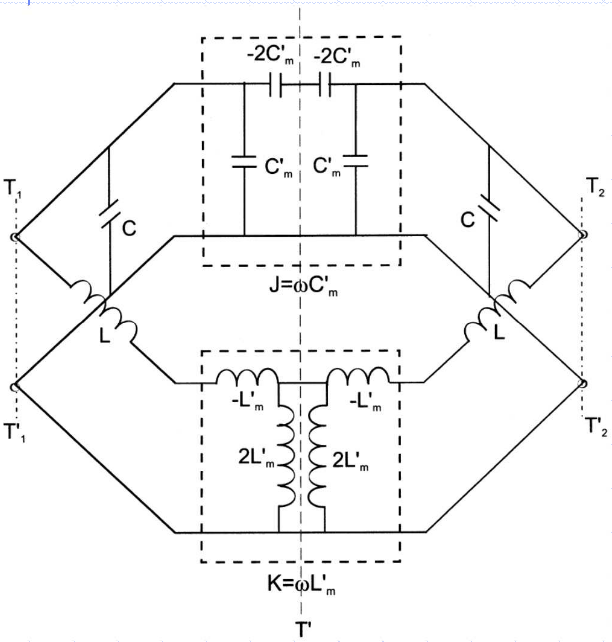
$$Y_{12} = Y_{21} = j\omega C'_m$$

Magnetic coupling:

$$Z_{11} = Z_{22} = j\omega L$$

$$Z_{12} = Z_{21} = j\omega L'_m$$

Synchronously Tuned Coupled-Resonator Circuit - Mixed Coupling (2/2)



at the reference plane T-T'

Electric wall:

$$f_e = \frac{1}{2\pi\sqrt{(L - L'_m)(C - C'_m)}}$$

Magnetic wall:

$$f_m = \frac{1}{2\pi\sqrt{(L + L'_m)(C + C'_m)}}$$

Mixed coupling coefficients can be obtained:

$$k_X = \frac{f_e^2 - f_m^2}{f_e^2 + f_m^2} = \frac{CL'_m + LC'_m}{LC + L'_m C'_m}$$

Assume $L'_m C'_m \ll LC$

$$k_X \approx \frac{L'_m}{L} + \frac{C'_m}{C} = k'_M + k'_E$$

The superposition of the magnetic and electric couplings!

Comment

Care should be taken for the mixed coupling because the superposition of both the magnetic and electric couplings can result in two opposite effects, either enhancing or canceling each other as mentioned before.

If we allow either the mutual inductance or the mutual capacitance in Figure 8.6(*b) to change sign, we will find that both couplings tend to cancel each other* out.

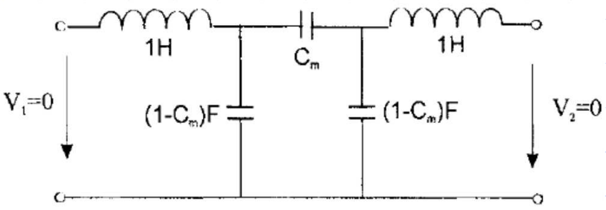
It should be remarked that for numerical computations, depending on the particular EM simulator used, as well as the coupling structure analyzed, it may sometimes be difficult to implement the electric wall, the magnetic wall, or even both in the simulation. This difficulty is more obvious for experiments.

The difficulty can be removed easily by analyzing or measuring the whole coupling structure instead of the half, and finding the natural resonant frequencies of two resonant peaks, observable from the resonant frequency response. It has been proved that the two natural resonant frequencies obtained in this way are *fe and fm [5]*.

Nature Frequencies

- It is sometimes difficult to apply the electric and magnetic wall to the coupled structure. However, the f_e and f_m are two the nature frequencies of the coupled structure.
- Consider the electric coupling circuit

$$\omega_o = 1/\sqrt{LC} = 1 \qquad z_o = \sqrt{L/C} = 1$$



Electric Coupling

$$Z_{11} = Z_{22} = \frac{A}{B}, \qquad A = 1 + \frac{1-C_m}{C_m} - \omega^2 \cdot \frac{(1-C_m)(1+C_m)}{C_m}$$

$$Z_{21} = Z_{12} = \frac{1}{B}, \qquad B = j\omega\frac{(1-C_m)(1+C_m)}{C_m}$$

- Imposing the boundary conditions $V_1 = V_2 = 0$ for nature resonance

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \Rightarrow \begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix} = 0$$

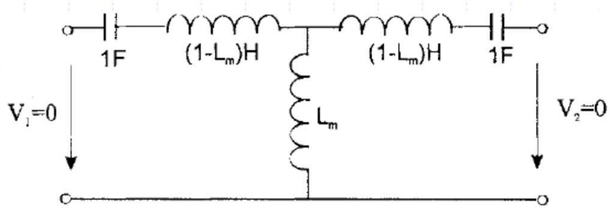
$$\Rightarrow Z_{11} \cdot Z_{22} - Z_{12} \cdot Z_{21} = 0$$

$$\Rightarrow \begin{aligned} \omega &= 2\pi \cdot f_e = 1/\sqrt{1+C_m} \\ \omega &= 2\pi \cdot f_m = 1/\sqrt{1-C_m} \end{aligned}$$

$\Rightarrow$ The two solved frequencies match with the results based on inserting E-wall and H-wall.

Nature Frequencies

- Magnetic coupling circuit



Magnetic Coupling

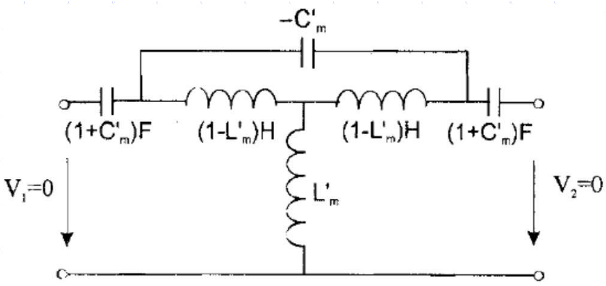
$$Z_{11} \cdot Z_{22} - Z_{12} \cdot Z_{21} = 0$$

$$\Rightarrow Z_{11} = Z_{22} = \frac{A}{B}, \qquad Z_{21} = Z_{12} = \frac{1}{B},$$

$$A = 1 + \frac{(1-L_m) - \frac{1}{\omega^2}}{L_m} \qquad B = \frac{1}{j\omega L_m}$$

$$\Rightarrow \begin{aligned} \omega &= 2\pi \cdot f_e = 1/\sqrt{1-L_m} \\ \omega &= 2\pi \cdot f_m = 1/\sqrt{1+L_m} \end{aligned}$$

- Mixed coupling circuit



Mixed Coupling

$$Z_{11} \cdot Z_{22} - Z_{12} \cdot Z_{21} = 0$$

$$\Rightarrow Z_{11} = Z_{22} = \frac{Z_{11e} + Z_{11o}}{2}, \qquad Z_{21} = Z_{12} = \frac{Z_{11e} - Z_{11o}}{2},$$

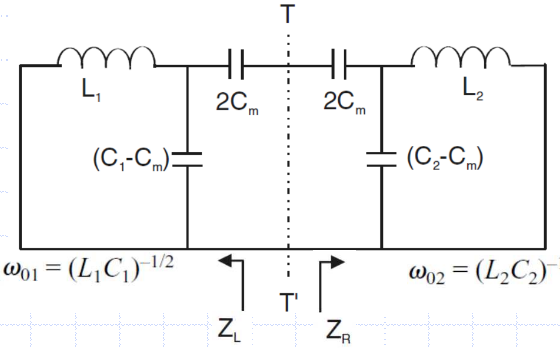
$$Z_{11e} = \frac{1 - \omega^2(1+L'_m)(1+C'_m)}{j\omega(1+C'_m)},$$

$$Z_{11o} = \frac{1 - \omega^2(1-L'_m)(1-C'_m)}{j\omega(1+C'_m)[1+2\omega^2 C'_m(1-L'_m)]}$$

$$\Rightarrow \begin{aligned} \omega &= 2\pi \cdot f_m = 1/\sqrt{(1-L'_m)(1-C'_m)} \\ \omega &= 2\pi \cdot f_e = 1/\sqrt{(1+L'_m)(1+C'_m)} \end{aligned}$$

Asynchronously Tuned Coupled-Resonator Circuit - Electric Coupling

- Electric coupling coefficients of coupled resonators



$\omega_{01} = (L_1C_1)^{-1/2}$ $\omega_{02} = (L_2C_2)^{-1/2}$

where Z_L and Z_R are the input impedances

at the reference plane T-T', the resonant condition

$$Z_L = -Z_R$$

$$\frac{1}{j\omega C_m} + \frac{j\omega L_1}{1-\omega^2 L_1(C_1-C_m)} + \frac{j\omega L_2}{1-\omega^2 L_2(C_2-C_m)} = 0$$

$$\omega^4(L_1L_2C_1C_2 - L_1L_2C_m^2) - \omega^2(L_1C_1+L_2C_2) + 1 = 0$$

- Two positive solutions:

$$\omega_{1,2} = \sqrt{\frac{(L_1C_1+L_2C_2) \pm \sqrt{(L_1C_1-L_2C_2)^2 + 4L_1L_2C_m^2}}{2(L_1L_2C_1C_2 - L_1L_2C_m^2)}}$$

define a parameter

$$K_E = \frac{\omega_2^2 - \omega_1^2}{\omega_2^2 + \omega_1^2}$$

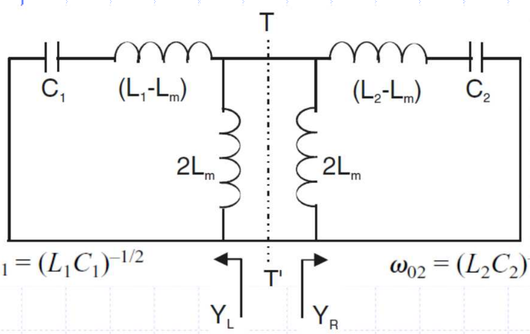
$$K_E^2 = \frac{C_m^2}{C_1C_2}\frac{4}{\left(\frac{\omega_{02}}{\omega_{01}} + \frac{\omega_{01}}{\omega_{02}}\right)^2} + \left(\frac{\omega_{02}^2 - \omega_{01}^2}{\omega_{02}^2 + \omega_{01}^2}\right)^2$$

- Electric coupling coefficients:

$$k_e = \frac{C_m}{\sqrt{C_1C_2}} = \pm\frac{1}{2}\left(\frac{\omega_{02}}{\omega_{01}} + \frac{\omega_{01}}{\omega_{02}}\right)\sqrt{\left(\frac{\omega_2^2 - \omega_1^2}{\omega_2^2 + \omega_1^2}\right)^2 - \left(\frac{\omega_{02}^2 - \omega_{01}^2}{\omega_{02}^2 + \omega_{01}^2}\right)^2}$$

Asynchronously Tuned Coupled-Resonator Circuit - Magnetic Coupling

- Magnetic coupling coefficients of coupled resonators



$\omega_{01} = (L_1C_1)^{-1/2}$ $\omega_{02} = (L_2C_2)^{-1/2}$

where Y_L and Y_R are the input admittance

at the reference plane T-T', the resonant condition

$$Y_L = -Y_R$$

$$\frac{1}{j\omega L_m} + \frac{j\omega C_1}{1-\omega^2 C_1(L_1-L_m)} + \frac{j\omega C_2}{1-\omega^2 C_2(L_2-L_m)} = 0$$

$$\omega^4(L_1L_2C_1C_2 - C_1C_2L_m^2) - \omega^2(L_1C_1+L_2C_2) + 1 = 0$$

- Two positive solutions:

$$\omega_{1,2} = \sqrt{\frac{(L_1C_1+L_2C_2) \pm \sqrt{(L_1C_1-L_2C_2)^2 + 4C_1C_2L_m^2}}{2(L_1L_2C_1C_2 - C_1C_2L_m^2)}}$$

define a parameter

$$K_M = \frac{\omega_2^2 - \omega_1^2}{\omega_2^2 + \omega_1^2}$$

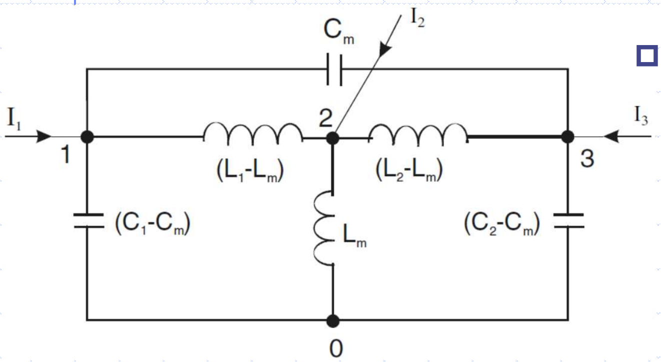
$$K_M^2 = \frac{L_m^2}{L_1L_2}\frac{4}{\left(\frac{\omega_{02}}{\omega_{01}} + \frac{\omega_{01}}{\omega_{02}}\right)^2} + \left(\frac{\omega_{02}^2 - \omega_{01}^2}{\omega_{02}^2 + \omega_{01}^2}\right)^2$$

- Magnetic coupling coefficients:

$$k_m = \frac{L_m}{\sqrt{L_1L_2}} = \pm\frac{1}{2}\left(\frac{\omega_{02}}{\omega_{01}} + \frac{\omega_{01}}{\omega_{02}}\right)\sqrt{\left(\frac{\omega_2^2 - \omega_1^2}{\omega_2^2 + \omega_1^2}\right)^2 - \left(\frac{\omega_{02}^2 - \omega_{01}^2}{\omega_{02}^2 + \omega_{01}^2}\right)^2}$$

Asynchronously Tuned Coupled-Resonator Circuit - Mixed Coupling (1/3)

- Mixed coupling coefficients of coupled resonators



- Define a nodal admittance matrix with a reference at node “0”

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$y_{11} = j\omega C_1 + \frac{1}{j\omega(L_1 - L_m)}$$

$$y_{12} = y_{21} = -\frac{1}{j\omega(L_1 - L_m)}$$

$$y_{13} = y_{31} = -j\omega C_m$$

$$y_{22} = \frac{1}{j\omega L_m} + \frac{1}{j\omega(L_1 - L_m)} + \frac{1}{j\omega(L_2 - L_m)}$$

$$y_{23} = y_{32} = -\frac{1}{j\omega(L_2 - L_m)}$$

$$y_{33} = j\omega C_2 + \frac{1}{j\omega(L_2 - L_m)}$$

Asynchronously Tuned Coupled-Resonator Circuit - Mixed Coupling (2/3)

- For resonant condition, it implies

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ for } \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \Rightarrow \begin{vmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{vmatrix} = 0$$

- To solve the determinant of the admittance matrix to be zero, it obtains

$$\omega^4(L_1C_1L_2C_2 - L_m^2C_1C_2 - L_1L_2C_m^2 + L_m^2C_m^2) - \omega^2(L_1C_1 + L_2C_2 - 2L_mC_m) + 1 = 0$$

- Two positive solutions are of interest

$$\omega_1 = \sqrt{\frac{\Re_B - \Re_C}{\Re_A}} \qquad \omega_2 = \sqrt{\frac{\Re_B + \Re_C}{\Re_A}}$$

with

$$\Re_A = 2(L_1C_1L_2C_2 - L_m^2C_1C_2 - L_1L_2C_m^2 + L_m^2C_m^2)$$

$$\Re_B = (L_1C_1 + L_2C_2 - 2L_mC_m)$$

$$\Re_C = \sqrt{\Re_B^2 - 2\Re_A}$$

Asynchronously Tuned Coupled-Resonator Circuit - Mixed Coupling (3/3)

- Define a parameter

$$K_X = \frac{\omega_2^2 - \omega_1^2}{\omega_2^2 + \omega_1^2}$$

- For narrowband applications, assume that

$$(L_1C_1 + L_2C_2) \gg L_mC_m \quad \text{and} \quad \frac{(L_1C_1 + L_2C_2)/2}{\sqrt{L_1C_1L_2C_2}} \approx 1$$

$$\Rightarrow K_X^2 = \frac{4L_1C_1L_2C_2}{(L_1C_1 + L_2C_2)^2} k_x^2 + \frac{(L_1C_1 - L_2C_2)^2}{(L_1C_1 + L_2C_2)^2}$$

$$\Rightarrow k_x^2 = \left(\frac{C_m^2}{C_1C_2} + \frac{L_m^2}{L_1L_2} - \frac{2L_mC_m}{\sqrt{L_1C_1L_2C_2}} \right) = (k_e - k_m)^2$$

- The mixed coupling coefficient is defined as

$$k_x = k_e - k_m = \pm \frac{1}{2} \left(\frac{\omega_{02}}{\omega_{01}} + \frac{\omega_{01}}{\omega_{02}} \right) \sqrt{ \left(\frac{\omega_2^2 - \omega_1^2}{\omega_2^2 + \omega_1^2} \right)^2 - \left(\frac{\omega_{02}^2 - \omega_{01}^2}{\omega_{02}^2 + \omega_{01}^2} \right)^2 }$$

General Formulation of Coupling Coefficient

- The formulas for extracting coupling coefficients on electric, magnetic, and mixed coupling of asynchronous tuned coupled resonators are all the same.

$$k = \pm \frac{1}{2} \left(\frac{f_{02}}{f_{01}} + \frac{f_{01}}{f_{02}} \right) \sqrt{ \left(\frac{f_{p2}^2 - f_{p1}^2}{f_{p2}^2 + f_{p1}^2} \right)^2 - \left(\frac{f_{02}^2 - f_{01}^2}{f_{02}^2 + f_{01}^2} \right)^2 } \quad \text{where} \quad \begin{aligned} f_{0i} &= \omega_{0i}/2\pi \\ f_{pi} &= \omega_i/2\pi \text{ for } i = 1, 2 \end{aligned}$$

- For synchronously tuned coupled resonators, it degenerates to

$$k = \pm \frac{f_{p2}^2 - f_{p1}^2}{f_{p2}^2 + f_{p1}^2} \qquad f_{p1} \text{ or } f_{p2} \text{ corresponds to either } f_e \text{ or } f_m$$

- As a result, the above definitions can be used to extract the coupling coefficient of any two coupled-resonator, regardless of whether the coupling is electric, magnetic, or mixed.
- The sign of the coupling coefficients depends on the physical coupling structure; nevertheless, for filter design, the meaning of positive/negative coupling is rather relative.

Comment

This means that if we refer to one particular coupling as the positive coupling, and then the negative coupling would imply that its phase response is opposite to that of the positive coupling.

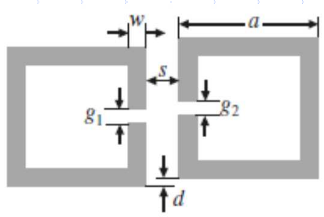
The phase response of a coupling may be found from the *S parameters* of its associated coupling structure.

Alternatively, the derivations in Section 8.2.1 have suggested another simple way to find whether the two coupling structures have the same signs or not.

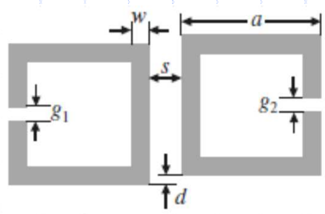
This can be done by applying either the electric or magnetic wall to find the *fe or fm of both the coupling structures. If the frequency* shifts of *fe or fm with respect to their individual uncoupled resonant frequencies are* in the same direction, the resultant coupling coefficients will have the same signs, if not the opposite signs.

Numerical Examples

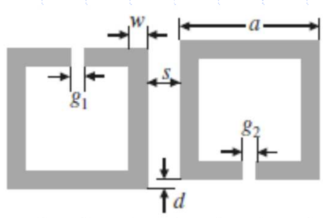
- Folded half-wavelength resonators (open-loop resonators)
  - Electric coupling



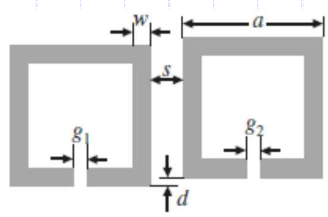
  - Magnetic coupling



  - Mixed coupling

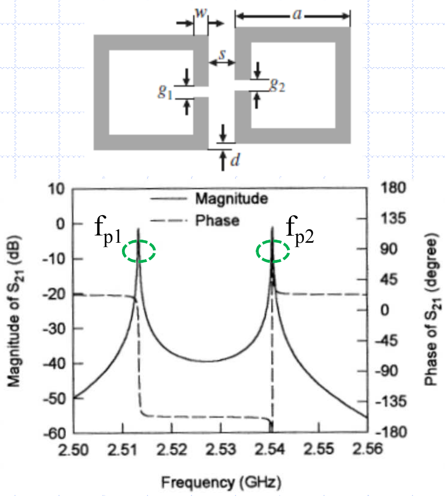


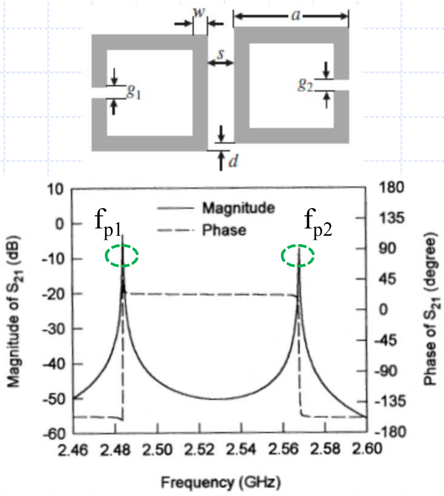
  - Mixed coupling



Extracting Coupling Coefficients - Synchronous Tuning

- Folded half-wavelength resonators (open-loop resonators)
  - Electric coupling
  - Magnetic coupling





Electric coupling:

$f_{p1} = 2513.3$ MHz $\quad f_{p2} = 2540.7$ MHz

$$k = \frac{f_{p2}^2 - f_{p1}^2}{f_{p2}^2 + f_{p1}^2} = 0.01084$$

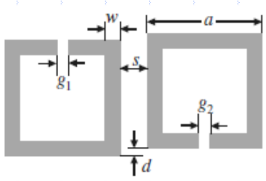
Magnetic coupling:

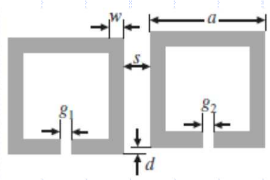
$f_{p1} = 2484.2$ MHz $\quad f_{p2} = 2567.9$ MHz

$$k = \frac{f_{p2}^2 - f_{p1}^2}{f_{p2}^2 + f_{p1}^2} = 0.03313$$

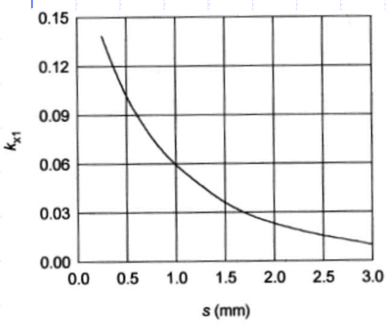
Extracting Coupling Coefficients - Synchronous Tuning

- Mixed coupling structures

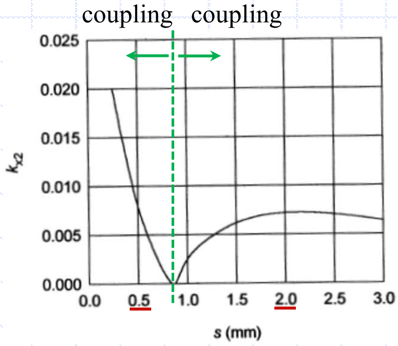




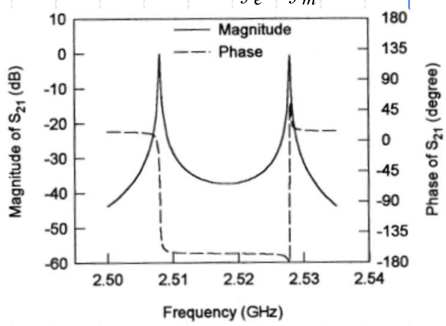
Electric coupling



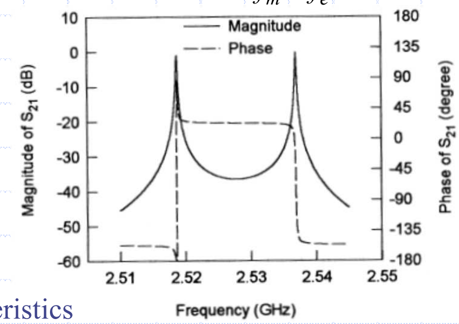
Electric coupling | Magnetic coupling



S=0.5 mm: $f_e < f_m$



S=2.0 mm: $f_m < f_e$



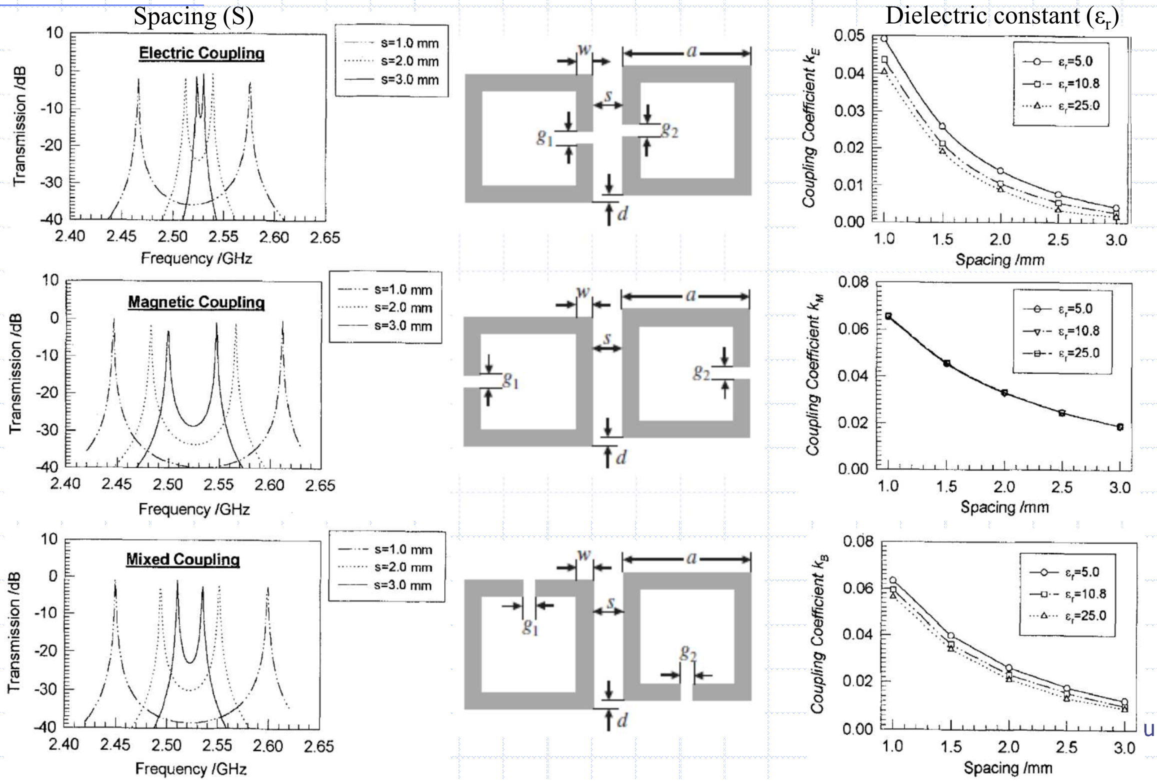
Note:

Two kinds of mixed coupling structure with different characteristics

Extracting Coupling Coefficients
- Synchronous Tuning (Parameters Sweep)

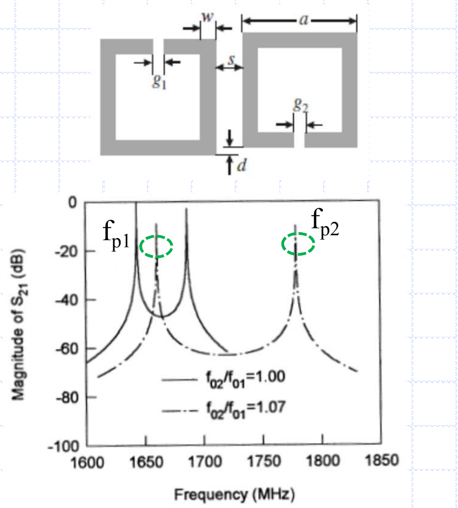
Spacing (S)

Dielectric constant (ε_r)



Extracting Coupling Coefficients
- Asynchronous Tuning

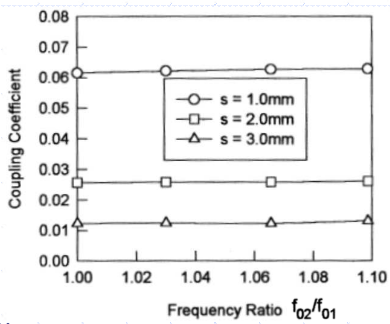
- Folded half-wavelength resonators ($g_1 \neq g_2 \Rightarrow f_{01} \neq f_{02}$)
  - Mixed coupling



$$k = \pm \frac{1}{2}\left(\frac{f_{02}}{f_{01}} + \frac{f_{01}}{f_{02}}\right)\sqrt{\left(\frac{f_{p2}^2 - f_{p1}^2}{f_{p2}^2 + f_{p1}^2}\right)^2 - \left(\frac{f_{02}^2 - f_{01}^2}{f_{02}^2 + f_{01}^2}\right)^2}$$

TABLE 8.1 Numerical results of the coupled microstrip open-loop resonators of Figure 8.15(*c*) for g_1 fixed by 0.4 mm

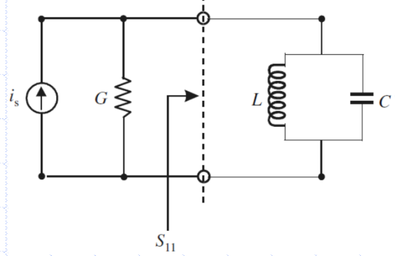
s (mm)	g_2 (mm)	f_{01}(MHz)	f_{02} (MHz)	f_{p1} = (MHz)	f_{p2} (MHz)
1	0.4	1664.7	1664.7	1613.2	1715.7
1	1.0	1664.7	1714.7	1631.2	1747.5
1	2.0	1664.7	1774.05	1642.0	1795.5
1	3.0	1664.7	1828.9	1647.5	1845.0
2	0.4	1664.7	1664.7	1643.4	1686.1
2	1.0	1664.7	1714.7	1656.6	1722.9
2	2.0	1664.7	1774.05	1660.4	1778.4
2	3.0	1664.7	1828.9	1661.7	1832.1
3	0.4	1664.7	1664.7	1654.5	1674.9
3	1.0	1664.7	1714.7	1662.6	1716.8
3	2.0	1664.7	1774.05	1663.7	1775.05
3	3.0	1664.7	1828.9	1664.0	1829.8



Note:
This type of coupled resonator shows robustness in asynchronous tuning (fabrication tolerance)

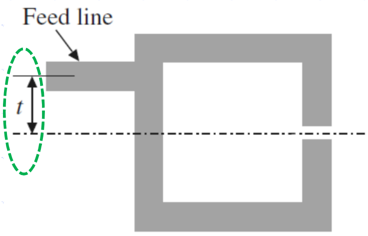
Formulation for Extracting External Quality Factor Q_e

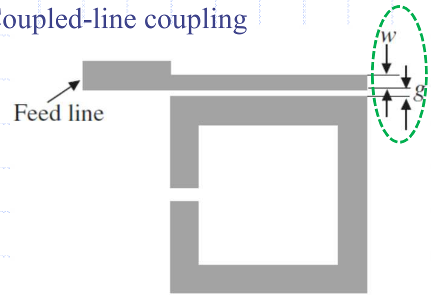
- External quality factor Q_e



$$Q_e = \frac{\omega_0 C}{G} = \frac{1}{G\omega_0 L}$$

- Two typical input/output coupling structures
  - Tapped-line coupling
  - Coupled-line coupling

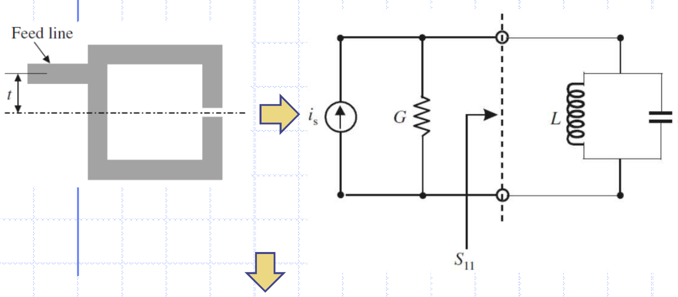




- Weaker coupling implies large external quality factor

External Quality Factor Q_e - Singly Loaded Resonator

- Extract the external quality factor from the frequency response



$$S_{11} = \frac{G - Y_{in}}{G + Y_{in}} = \frac{1 - Y_{in}/G}{1 + Y_{in}/G}$$

In the vicinity of resonance

$$(\omega^2 - \omega_0^2)/\omega \approx 2\Delta\omega$$

$$Y_{in} = j\omega C + \frac{1}{j\omega L} = j\omega_0 C\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

$$Y_{in} = j\omega_0 C \cdot \frac{2\Delta\omega}{\omega_0}$$

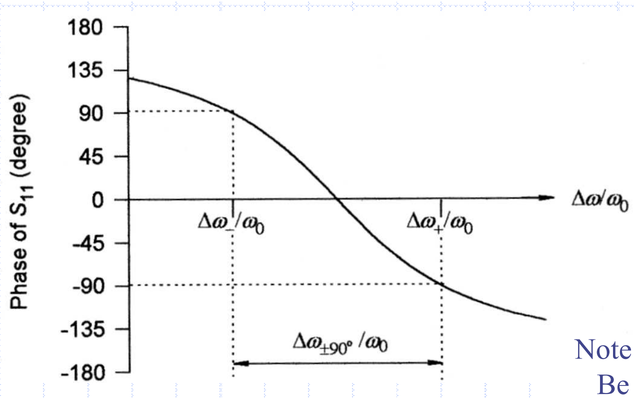
$$Q_e = \omega_0 C/G$$

$$S_{11} = \frac{1 - jQ_e \cdot (2\Delta\omega/\omega_0)}{1 + jQ_e \cdot (2\Delta\omega/\omega_0)} = \mp 1$$

$$\angle S11 = 90^o \qquad 2Q_e \frac{\Delta\omega_{\mp}}{\omega_0} = \mp 1$$

$$\Delta\omega_{\pm 90^\circ} = \Delta\omega_+ - \Delta\omega_- = \frac{\omega_0}{Q_e}$$

$$Q_e = \frac{\omega_0}{\Delta\omega_{\pm 90^\circ}}$$



Note:
Be careful about the reference plane of physical structure

External Quality Factor Q_e – Singly Loaded Resonator

- Extract the external quality factor from the group delay of S_{11}

$$S_{11} = \frac{1 - jQ_e \cdot (2\Delta\omega/\omega_0)}{1 + jQ_e \cdot (2\Delta\omega/\omega_0)} \Rightarrow S_{11} = e^{-j2\phi} \quad \text{where} \quad \phi = \tan^{-1}\left(\frac{2Q_e\Delta\omega}{\omega_0}\right)$$

  - The group delay of S_{11} is given

$$\tau_{S_{11}}(\omega) = -\frac{\partial(-2\phi)}{\partial\omega} = \frac{4Q_e}{\omega_0} \cdot \frac{1}{1 + (2Q_e\Delta\omega/\omega_0)^2}$$

  - At the resonance $\Delta\omega = 0$, it reaches the maximum value

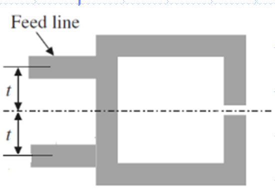
$$\tau_{S_{11}}(\omega_0) = \frac{4Q_e}{\omega_0} \Rightarrow Q_e = \frac{\omega_0 \cdot \tau_{S_{11}}(\omega_0)}{4}$$

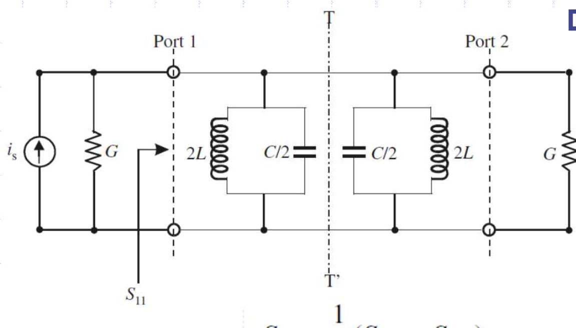
- For the physical structure, the reference plane of S_{11} may not coincide with that of equivalent circuit, an extra group delay should be added.

- Nonetheless, the resonant frequency ω_0 still can be determinable from the simulated frequency response of group delay.

External Quality Factor Q_e - Doubly Loaded Resonator

- Extract the external quality factor for symmetrical resonator





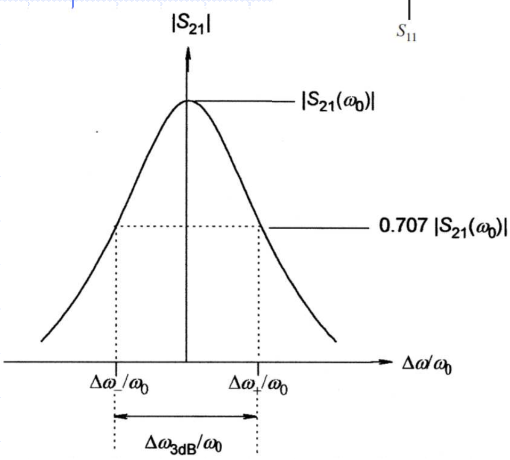
- Symmetrical network analysis

$$Y_{ino} = \infty$$

$$S_{11o} = \frac{G - Y_{ino}}{G + Y_{ino}} = -1$$

$$Y_{ine} = j\omega_0 C\Delta\omega/\omega_0$$

$$S_{11e} = \frac{G - Y_{ine}}{G + Y_{ine}} = \frac{1 - jQ_e\Delta\omega/\omega_0}{1 + jQ_e\Delta\omega/\omega_0}$$



$$S_{21} = \frac{1}{2}(S_{11e} - S_{11o}) = \frac{1}{1 + jQ_e\Delta\omega/\omega_0}$$

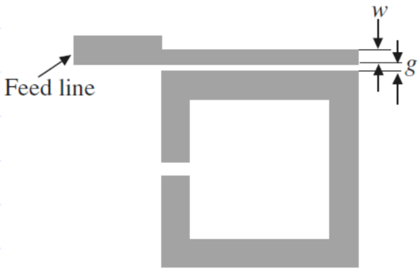
$$|S_{21}| = \frac{1}{\sqrt{1 + (Q_e\Delta\omega/\omega_0)^2}} \Rightarrow \begin{cases} \Delta\omega = 0 \Rightarrow |S_{21}(\omega_0)| = 1 \\ Q_e\dfrac{\Delta\omega_{\pm}}{\omega_0} = \pm 1 \Rightarrow |S_{21}(\omega_0)| = 0.707 \end{cases}$$

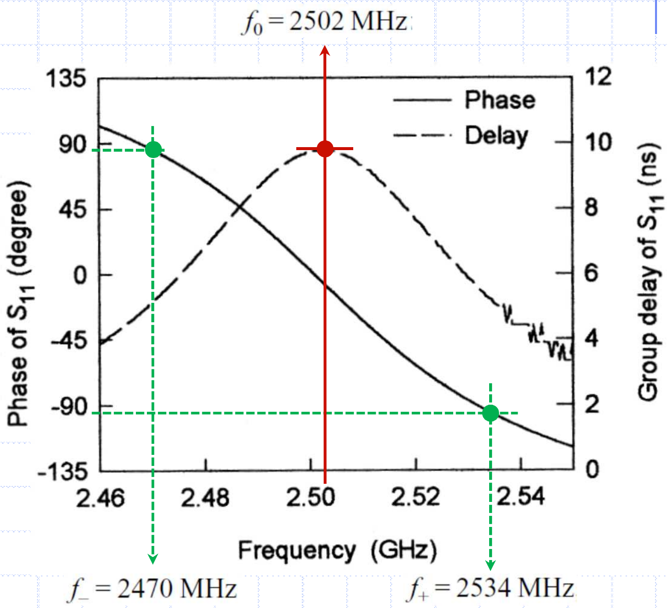
$$\Rightarrow \Delta\omega_{3\text{ dB}} = \Delta\omega_+ - \Delta\omega_- = \frac{\omega_0}{(Q_e/2)}$$

$$\Rightarrow \frac{Q_e}{2} = \frac{\omega_0}{\Delta\omega_{3\text{ dB}}}$$

Numerical Examples

- Extracting external quality factor using coupled-line coupling

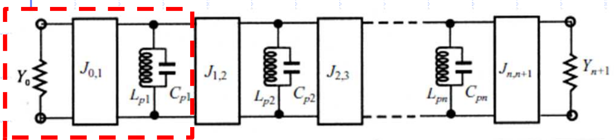




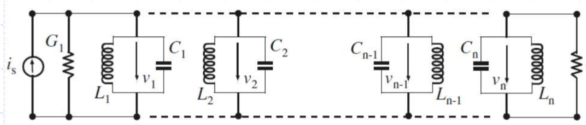
- From the phase response:
  - $f_0 = 2502$ MHz, $\phi_0 = -4.214^o \neq 0$
  - $f_- = 2470$ MHz, $\phi_- = 90^o - 4.214^o = 85.786^o$
  - $f_+ = 2534$ MHz, $\phi_+ = -90^o - 4.214^o = -94.214^o$
  - $\Delta f_{\pm 90^o} = f_+ - f_- = 64$ MHz
  - $Q_e = f_0/\Delta f_{\pm 90^o} = 2502/64 = 39$

Example
- Hairpin-Line Bandpass Filters

- Design a five order microstrip hairpin bandpass filter in chebyshev prototype with passband ripple of 0.1 dB. This filter have $FBW = 0.2$ at $f_0 = 2$ GHz.
  - Relationship between immitance inverter values from the conventional bandpass filter and the external quality factor and coupling coefficients from coupled-resonator filter

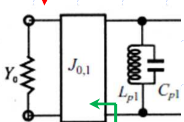


$$J_{0,1} = \sqrt{\frac{Y_0 FBW \omega_0 C_{p1}}{\Omega_c g_0 g_1}},\quad J_{i,i+1} = \frac{FBW\omega_0}{\Omega_c}\sqrt{\frac{C_{pi}C_{p(i+1)}}{g_i g_{i+1}}}\Bigg|_{i=1\text{ to }n-1},\quad J_{n,n+1} = \sqrt{\frac{FBW\omega_0 C_{pn} Y_{n+1}}{\Omega_c g_n g_{n+1}}}$$



$$Q_{ei} = \frac{\omega_0 C_1}{G_1} \qquad M_{i,i+1} = \frac{C_{i,i+1}}{\sqrt{C_i C_{i+1}}}$$

- For external quality factor



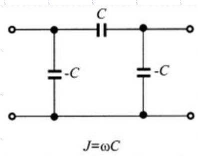
$$G_1 = \frac{J_{0,1}^2}{Y_0}$$

$$Q_{ei} = \frac{\omega_0 C_1}{G_1} = \frac{\omega_0 C_{p1}}{J_{0,1}^2/Y_0}$$

$$J_{0,1} = \sqrt{\frac{Y_0 FBW \omega_0 C_{p1}}{\Omega_c g_0 g_1}} \xRightarrow{\Omega_c = 1} \frac{\omega_0 C_{p1}}{J_{0,1}^2/Y_0} = \frac{g_0 g_1}{FBW}$$

$$Q_{ei} = \frac{\omega_0 C_1}{G_1} = \frac{g_0 g_1}{FBW}$$

- For coupling coefficients



$$M_{i,i+1} = \frac{C_{i,i+1}}{\sqrt{C_i C_{i+1}}} = \frac{\omega_0 C_{i,i+1}}{\omega_0\sqrt{C_i C_{i+1}}} = \frac{J_{i,i+1}}{\omega_0\sqrt{C_i C_{i+1}}}$$

$$J_{i,i+1} = \frac{FBW\omega_0}{\Omega_c}\sqrt{\frac{C_{p1}C_{p(i+1)}}{g_i g_{i+1}}} \xRightarrow{\Omega_c = 1} \frac{J_{i,i+1}}{\omega_0\sqrt{C_{p1}C_{p(i+1)}}} = \frac{FBW}{\sqrt{g_i g_{i+1}}}$$

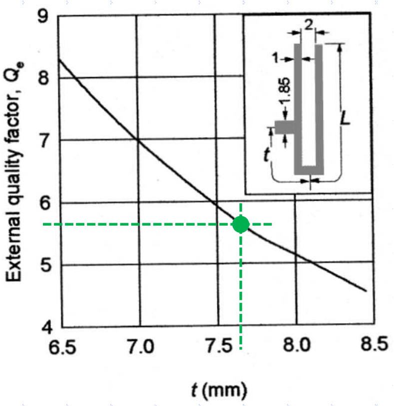
$$M_{i,i+1} = \frac{C_{i,i+1}}{\sqrt{C_i C_{i+1}}} = \frac{FBW}{\sqrt{g_i g_{i+1}}}$$

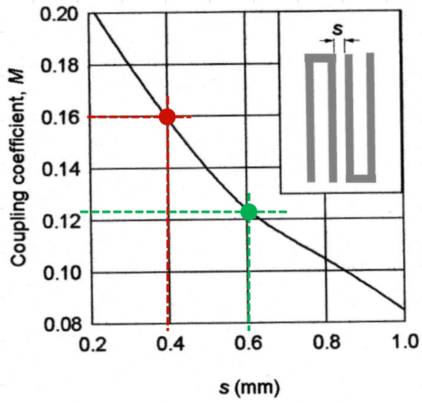
Example
- Hairpin-Line Bandpass Filters

- Design a five order microstrip hairpin bandpass filter in chebyshev prototype with passband ripple of 0.1 dB. This filter have $FBW = 0.2$ at $f_0 = 2$ GHz.
  - The external quality factor and coupling coefficients are given as below to achieve the frequency response

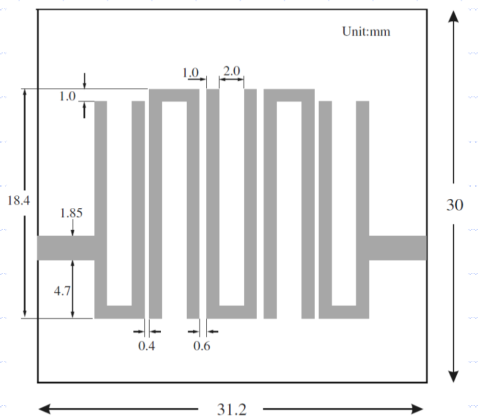
$$Q_{e1} = Q_{e5} = 5.734$$
$$M_{1,2} = M_{4,5} = 0.160$$
$$M_{2,3} = M_{3,4} = 0.122$$

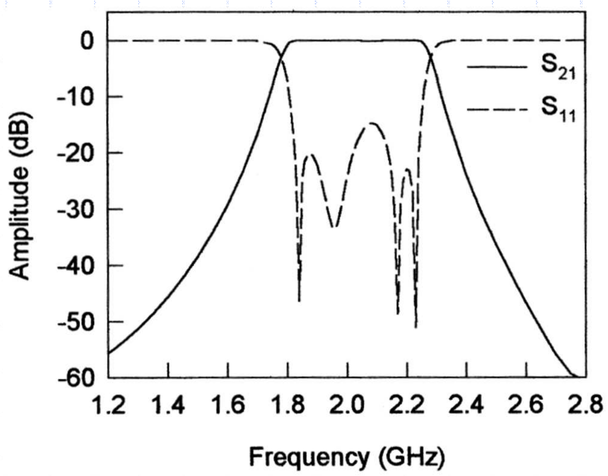
  - Using EM simulator to find the required values





Example
- Hairpin-Line Bandpass Filters





The microstrip is designed on a substrate with a dielectric constant of 6.15 and a thickness of 1.27 mm