Transmission-Line for Communication

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Introduction

Chapter 6: transient state in time-domain, (digital circuits)
Chapter 7: steady state in frequency domain, (communication circuits)

In the steady state, the situation is equivalent to the superposition of one (+) wave, which is the sum of all the transient (+) waves, and one (-) wave, which is the sum of all the transient (-) waves.

Thus, the general solutions for the line voltage and line current in the sinusoidal steady state are superposition of voltages and currents, respectively, of sinusoidal (+) and (-) waves.
Introduction

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- 7.2 Line Terminated by Arbitrary Load
- 7.3 Transmission-Line Matching
- 7.4 The Smith Chart: 1. Basic Procedures
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7.1 Short-Circuited Line

General solution in the sinusoidal steady state

The general solutions for the line voltage and line current in the sinusoidal steady state

\[ V(z,t) = A \cos \left[ \omega \left( t - \frac{z}{v_p} \right) + \theta \right] + B \cos \left[ \omega \left( t + \frac{z}{v_p} \right) + \phi \right] \]

\[ I(z,t) = \frac{1}{Z_0} \left\{ A \cos \left[ \omega \left( t - \frac{z}{v_p} \right) + \theta \right] - B \cos \left[ \omega \left( t + \frac{z}{v_p} \right) + \phi \right] \right\} \]

The corresponding expressions for the phasor line voltage and phasor line current

\[ \overline{V}(z) = \overline{V}^+ e^{-j\beta z} + \overline{V}^- e^{j\beta z} \]

\[ \overline{I}(z) = \frac{1}{Z_0} \left( \overline{V}^+ e^{-j\beta z} - \overline{V}^- e^{j\beta z} \right) \]

where \( \overline{V}^+ = Ae^{j\beta} \) and \( \overline{V}^- = Be^{j\beta} \), \( \beta = \omega/v_p \)

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7.1 Short-Circuited Line
General solution in the sinusoidal steady state

It is convenient to use a distance variable $d$ that increases as we go from the load toward the generator as opposed to $z$, which increased from the generator to the load.

$$\bar{V}(d) = \bar{V}^+ e^{j\beta d} + \bar{V}^- e^{-j\beta d}$$

$$\bar{I}(d) = \frac{1}{Z_0} \left( \bar{V}^+ e^{j\beta d} - \bar{V}^- e^{-j\beta d} \right)$$

We wish to determine the characteristics of the waves satisfying the boundary condition at the short circuit.
7.1 Short-Circuited Line

General solution in the sinusoidal steady state

\[ \bar{V}(0) = 0 \quad \rightarrow \quad \bar{V}(0) = \bar{V}^+ e^{j\beta(0)} + \bar{V}^- e^{-j\beta(0)} = 0 \quad \rightarrow \quad \bar{V} = -\bar{V} \]

Particular solutions for the complex voltage and current

\[ \bar{V}(d) = \bar{V}^+ e^{j\beta d} - \bar{V}^- e^{-j\beta d} = 2j\bar{V}^+ \sin \beta d \]

\[ \bar{I}(d) = \frac{1}{Z_0} \left( \bar{V}^+ e^{j\beta d} - \bar{V}^- e^{-j\beta d} \right) = 2\frac{\bar{V}^+}{Z_0} \cos \beta d \]

The real voltage and current are then given by

\[ V(d, t) = \text{Re} \left[ \bar{V}(d) e^{j\alpha t} \right] \]

\[ I(d, t) = \text{Re} \left[ \frac{\bar{I}(d) e^{j\alpha t}}{Z_0} \right] = \text{Re} \left( 2 \frac{\bar{V}^+}{Z_0} e^{j\theta} \cos \beta d e^{j\alpha t} \right) = 2 \frac{\bar{V}^+}{Z_0} \cos \beta d \cos(\alpha t + \theta) \]
7.1 Short-Circuited Line

General solution in the sinusoidal steady state

The instantaneous power flow down the line is given by

\[ P(d, t) = V(d, t)I(d, t) \]

\[ = -\frac{4|V|^2}{Z_0} \sin \beta d \cos \beta d \sin(\omega t + \theta) \cos(\omega t + \theta) \]

\[ = -\frac{|V|^2}{Z_0} \sin 2\beta d \sin 2(\omega t + \theta) \]

Why do we call them the standing wave?
7.1 Short-Circuited Line

Standing wave

Thus, the phenomenon on the short-circuited line is one in which the voltage, current, and power flow oscillate sinusoidally with time with different amplitudes at different locations on the line, unlike in the case of traveling waves in which a given point on the waveform progresses in distance with time.

Since there is no feeling of wave motion down the line, these waves are known as standing waves.

In particular, they represent complete standing waves in view of the zero amplitudes of the voltage, current, and power flow at certain locations on the line. Complete standing waves are the result of (+) and (-) traveling waves of equal amplitudes.

Is there any power flow at any location $d$ on the line?
7.1 Short-Circuited Line

Standing wave

Although there is instantaneous power flow at values of $d$ between the voltage and current nodes, there is no time-average power flow for any value of $d$.

\[
\langle P \rangle = \frac{1}{T} \int_{t=0}^{T} P(d, t) \, dt = \frac{\omega}{2\pi} \int_{t=0}^{2\pi/\omega} P(d, t) \, dt
\]

\[
= \frac{\omega}{2\pi} \frac{|V|^2}{Z_0} \sin 2\beta d \int_{t=0}^{2\pi/\omega} \sin 2(\omega t + \theta) \, dt
\]

\[
= 0
\]
7.1 Short-Circuited Line

Standing wave patterns

Take the amplitude

\[ |V(d)| = 2 |V^+| \sin \beta d = 2 |V^+| \sin \frac{2\pi}{\lambda} d \]

\[ |I(d)| = \frac{2 |V^+|}{Z_0} \cos \beta d = \frac{2 |V^+|}{Z_0} \cos \frac{2\pi}{\lambda} d \]

They are the patterns of line voltage and line current one would obtain by connecting an ac voltmeter between the conductors of the line and an ac ammeter in series with one of the conductors of the line and observing their readings at various points along the line.

Standing-wave patterns should not be misinterpreted as the voltage and current remaining constant with time at a given point. On the other hand, the voltage and current at every point on the line vary sinusoidally with time.
7.1 Short-Circuited Line

Natural Oscillations

Since there is no power flow across a voltage node or a current node of the standing-wave patterns, a constant amount of total energy is locked up in every section between two such adjacent nodes with exchange of energy taking place between the electric and magnetic fields.

Electrical length of a line: its length in terms of wavelength, which is depends on the frequency.

Let us now consider a line of length $L$, one end of which is open-circuited and the other end short-circuited, and assume that some energy is stored in this line.

What are all the possible standing-wave patterns on this line?
7.1 Short-Circuited Line

Natural Oscillation

We note that the voltage across the short circuit must always be zero, and, hence, the current there must have maximum amplitude. Similarly, the current at the open-circuited end must always be zero, and, hence, the voltage there must have maximum amplitude. (boundary condition)

We also know that the standing-wave patterns are sinusoidal with the distance between successive nodes corresponding to a half sine wave.

Thus, the least possible variation is a quarter cycle of a sine waveform.

\[ L = \frac{\lambda}{4} \]

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7.1 Short-Circuited Line

Natural Oscillation

Other possible patterns

\[ l = \frac{(2n-1)\lambda_n}{4} \quad n = 1, 2, 3, \ldots \]

\[ f_n = \frac{v_p}{\lambda_n} = \frac{(2n-1)v_p}{4l} \quad n = 1, 2, 3, \ldots \]
7.1 Short-Circuited Line

Natural Oscillation

\[ f_n = \frac{v_p}{\lambda_n} = \frac{(2n-1)v_p}{4l} \quad n = 1, 2, 3, \ldots \]

These frequencies are known as the **natural frequencies of oscillation**.

The standing-wave patterns are said to correspond to the different natural modes of oscillation. The lowest frequency (corresponding to the longest wavelength) is known as the **fundamental frequency of oscillation**, and the corresponding mode is known as the **fundamental mode**. The quantity \( n \) is called the **mode number**.

In the most general case of non-sinusoidal voltage and current distributions on the line, the situation corresponds to the **superposition of some or all of the infinite number of natural modes**.
7.1 Short-Circuited Line

Input impedance

We define the ratio of phasor line voltage and the phasor line current as the line impedance, $Z(d)$, at that point seen looking toward the short circuit.

$$\bar{Z}(d) = \frac{\bar{V}(d)}{I(d)} = \frac{2j\bar{V}^+ \sin \beta d}{2(\bar{V}^+ / Z_0) \cos \beta d} = jZ_0 \tan \beta d$$

$$\bar{Z}_{in} = jZ_0 \tan \beta l = jZ_0 \tan \frac{2\pi f}{\nu_p} l$$

The input impedance of the short-circuited line is purely reactive.

As the frequency is varied from a low value upward, the input reactance changes from inductive to capacitive and back to inductive, and so on.

The input reactance is zero for values of frequency equal to multiples of $\nu_p / 2l$. These are the frequencies for which $l$ is equal to multiples of $\lambda / 2$ so that the line voltage is zero at the input and hence the input sees a short circuit.

The input reactance is infinity for values of frequency equal to odd multiples of $\nu_p / 4l$. These are the frequencies for which $l$ is equal to odd multiples of $\lambda / 4$ so that the line current is zero at the input and hence the input sees an open circuit.
7.1 Short-Circuited Line

Input impedance

These properties of the input impedance of a short-circuited line (and, similarly, of an open-circuited line) have several applications.

- *Determination of the location of a short circuit (or open circuit) in a line.*

Since the difference between a pair of consecutive frequencies for which the input reactance values are zero and infinity is $v_p / 4l$

- *Construction of resonant circuits at microwave frequencies.*

The principle behind this lies in the fact that the input reactance of a short-circuited line of a given length can be inductive or capacitive, depending on the frequency, and hence, two short-circuited lines connected together form a resonant system.

How to derive characteristic equation for the resonant frequencies of such a system?

![Diagram of short-circuited line with input impedance and current arrows](image)
7.1 Short-Circuited Line

Input impedance

B.C. at the junction

\[ \overline{V}_1 = \overline{V}_2 \land \overline{I}_1 + \overline{I}_2 = 0 \land \frac{\overline{I}_1}{\overline{V}_1} + \frac{\overline{I}_2}{\overline{V}_2} = 0 \rightarrow \overline{Y}_1 + \overline{Y}_2 = 0 \]

\[
\overline{Y}_1 = \frac{1}{Z_1} = \frac{1}{jZ_{01} \tan \beta_1 l_1} = \frac{1}{jZ_{01} \tan(2\pi f / v_{p1})l_1}
\]
\[
\overline{Y}_2 = \frac{1}{Z_2} = \frac{1}{jZ_{02} \tan \beta_2 l_2} = \frac{1}{jZ_{02} \tan(2\pi f / v_{p2})l_2}
\]

characteristic equation for the resonance frequency

\[ Z_{01} \tan \left( \frac{2\pi f}{v_{p1}} \right) l_1 + Z_{02} \tan \left( \frac{2\pi f}{v_{p2}} \right) l_2 = 0 \]
7.2 LINE TERMINATED BY ARBITRARY LOAD

\[
\bar{I}(d) = \frac{1}{Z_0} (\bar{V}^+ e^{j\beta d} - \bar{V}^- e^{-j\beta d})
\]

\[
\bar{V}(d) = \bar{V}^+ e^{j\beta d} + \bar{V}^- e^{-j\beta d}
\]

B. C. on load

\[
\bar{V}(0) = \bar{Z}_R \bar{I}(0)
\]

\[
\bar{V}^+ + \bar{V}^- = \frac{\bar{Z}_R}{Z_0} (\bar{V}^+ - \bar{V}^-) \quad \Rightarrow \quad \bar{V}^- = \bar{V}^+ \frac{\bar{Z}_R - Z_0}{\bar{Z}_R + Z_0}
\]

\[
\bar{I}_R = \frac{\bar{V}}{\bar{V}^+} = \frac{\bar{Z}_R - Z_0}{\bar{Z}_R + Z_0}
\]

Voltage reflection coefficient at the load

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7.2 Line Terminated by Arbitrary Load

**Generalized reflection coefficient**

The solutions for \( V(d) \) and \( I(d) \) can then be written as

\[
\bar{V}(d) = \bar{V}^+ e^{j\beta d} + \bar{\Gamma}_R \bar{V}^+ e^{-j\beta d} \quad \bar{I}(d) = \frac{1}{Z_0} (\bar{V}^+ e^{j\beta d} - \bar{\Gamma}_R \bar{V}^+ e^{-j\beta d})
\]

**Generalized voltage reflection coefficient**, the voltage reflection coefficient at any value of \( d \), as the ratio of the reflected wave voltage to the incident wave voltage at that value of \( d \).

\[
\bar{\Gamma}(d) = \frac{\bar{\Gamma}_R \bar{V}^+ e^{-j\beta d}}{\bar{V}^+ e^{j\beta d}} = \bar{\Gamma}_R e^{-j2\beta d}
\]

\[
|\bar{\Gamma}(d)| = |\bar{\Gamma}_R| |e^{-j2\beta d}| = |\bar{\Gamma}_R| \quad \angle \bar{\Gamma}(d) = \angle \bar{\Gamma}_R + \angle e^{-j2\beta d} = \theta - 2\beta d
\]

\[
\bar{V}(d) = \bar{V}^+ e^{j\beta d} (1 + \bar{\Gamma}_R e^{-j2\beta d}) = \bar{V}^+ e^{j\beta d} \left[ 1 + \bar{\Gamma}(d) \right]
\]

\[
\bar{I}(d) = \frac{\bar{V}^+}{Z_0} e^{j\beta d} (1 - \bar{\Gamma}_R e^{-j2\beta d}) = \frac{\bar{V}^+}{Z_0} e^{j\beta d} \left[ 1 - \bar{\Gamma}(d) \right]
\]
7.2 Line Terminated by Arbitrary Load

*standing wave pattern*

To study the standing-wave patterns

\[
|\overline{V}(d)| = |\overline{V^+}| |e^{j\beta d}| \left|1 + \Gamma(d)\right|
\]

\[
= |\overline{V^+}| \left|1 + \Gamma_R e^{-j2\beta d}\right|
\]

\[
|\overline{I}(d)| = \left|\frac{\overline{V^+}}{Z_0}\right| e^{j\beta d} \left|1 - \Gamma(d)\right|
\]

\[
= \left|\frac{\overline{V^+}}{Z_0}\right| \left|1 - \Gamma_R e^{-j2\beta d}\right|
\]

\(\overline{V^+}\) is simply a constant, determined by the boundary condition at the source end.
7.2 Line Terminated by Arbitrary Load

*standing wave pattern*

From these constructions and assuming $-\pi \leq \theta \leq \pi$, we note the following facts:

1. Point $A$ lies along the positive real axis and point $B$ lies along the negative real axis for $(\theta - 2\beta d) = 0, -2\pi, -4\pi, -6\pi, \ldots$, or $d = (\lambda/4\pi)(\theta + 2\pi n)$, where $n = 0, 1, 2, 3, \ldots$. Hence, at these values of $d$, the voltage amplitude is maximum and equal to $|V^+| (1 + |\Gamma_R|)$, whereas the current amplitude is minimum and equal to $(|V^+|/Z_0)(1 - |\Gamma_R|)$. The voltage and current are in phase.
7.2 Line Terminated by Arbitrary Load

**standing wave pattern**

2. Point $A$ lies along the negative real axis and point $B$ lies along the positive real axis for $(\theta - 2\beta d) = -\pi, -3\pi, -5\pi, -7\pi, \ldots$, or $d = (\lambda/4\pi)(\theta + (2n - 1)\pi)$, where $n = 1, 2, 3, 4, \ldots$. Hence, at these values of $d$, the voltage amplitude is minimum and equal to $|V^+|(1 - |\Gamma_R|)$, whereas the current amplitude is maximum and equal to $(|V^+|/Z_0)(1 + |\Gamma_R|)$. The voltage and current are in phase.
7.2 Line Terminated by Arbitrary Load

*standing wave pattern*

3. Between maxima and minima, the voltage and current vary in accordance with the lengths of the line joining \((-1, 0)\) to the points \(A\) and \(B\), respectively, as they move around the circles. These variations are not sinusoidal with distance. The variations near the minima are sharper than are those near the maxima; hence, the minima can be located more accurately than can the maxima. Also, the voltage and current are not in phase.
7.2 Line Terminated by Arbitrary Load

*standing wave parameters (three parameters)*

1. The standing-wave ratio, abbreviated as SWR.

\[
\text{SWR} = \frac{V_{\text{max}}}{V_{\text{min}}} = \frac{|\overline{V}^+| (1 + |\Gamma_R|)}{|\overline{V}^+| (1 - |\Gamma_R|)} = \frac{1 + |\Gamma_R|}{1 - |\Gamma_R|}
\]

also

\[
\frac{I_{\text{max}}}{I_{\text{min}}} = \frac{(|\overline{V}^+|/Z_0) (1 + |\Gamma_R|)}{(|\overline{V}^+|/Z_0) (1 - |\Gamma_R|)} = \frac{1 + |\Gamma_R|}{1 - |\Gamma_R|}
\]

The SWR is a measure of standing waves on the line. It is an easily measurable parameter.

(a) For \( |\Gamma_R| = 0 \), SWR = 1

(b) For \( |\Gamma_R| = 1 \), SWR = \( \infty \)
7.2 Line Terminated by Arbitrary Load

**standing wave parameters (three parameters)**

2. **The distance of the first voltage minimum from the load, denoted by** $d_{\text{min}}$.

   The voltage minimum nearest to the load occurs when the phase angle of
   $\Gamma(d) = \Gamma_R e^{-j2\beta d}$ is equal to $-\pi$, that is, for $(\theta - 2\beta d)$ equal to $-\pi$. Thus,

   $d_{\text{min}} = \frac{\theta + \pi}{2\beta} = \frac{\lambda}{4\pi} (\theta + \pi)$

   $\bar{Z}_R > Z_0$ $\implies$ $\theta = 0$ $\implies$ $d_{\text{min}} = \lambda/4$

   $\bar{Z}_R < Z_0$ $\implies$ $\theta = -\pi$ $\implies$ $d_{\text{min}} = 0$

   Physical meaning?

3. **The wavelength** $\lambda$. Since the distance between successive voltage minima
   is equal to $\lambda/2$, the wavelength is twice the distance between successive
   voltage minima.
7.2 Line Terminated by Arbitrary Load

Determination of unknown load impedance

Conversely to the computation of standing-wave parameters for a given load impedance, an unknown load impedance can be determined from standing-wave measurements on a line of known characteristic impedance.

\[
|\bar{\Gamma}_R| = \frac{\text{SWR} - 1}{\text{SWR} + 1}
\]

\[
\theta = \frac{4\pi d_{\text{min}}}{\lambda} - \pi
\]

\[
\bar{Z}_R = Z_0 \frac{1 + \bar{\Gamma}_R}{1 - \bar{\Gamma}_R}
\]

How to get the SWR?
7.2 Line Terminated by Arbitrary Load

Slotted-line measurements

A traditional method of performing standing-wave measurements in the laboratory is by using a slotted line.

The slotted line is essentially a rigid coaxial line with air dielectric and having a length of about 1 meter (or at least a half wavelength long). The center conductor is supported by dielectric inserts.

A narrow longitudinal slot is cut in the outer conductor,

Width of the slot is so small that it has negligible influence on the current flow on the outer conductor.

A probe of small length intercepts a portion of the electric field between the inner and outer conductors, and a small voltage proportional to the line voltage at the probe’s location is developed between the probe and the outer conductor.

Since the SWR is the ratio of to the quantity of interest is the ratio of the two readings rather than the absolute values of the readings themselves. Therefore, absolute calibration of the detector is not required.
7.2 Line Terminated by Arbitrary Load

Slotted-line measurements

Since it is not always possible to measure the distances of the standing wave pattern minima from the location of the load, the following procedure is employed.

First, the line is terminated by a short circuit in the place of the load. One of the nulls in the resulting standing-wave pattern is taken as the reference point, as shown in Fig. 7.12(a). This establishes that the location of the load is an integral multiple of half-wavelengths from the reference point.

Next, the short circuit is removed and the load is connected. The voltage minimum then shifts away from the reference point, as shown in Fig. 7.12(b). By measuring this shift, either away from the load or toward the load, the value of can be established.

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7.2 Line Terminated by Arbitrary Load

**Line Impedance**

The impedance at any value of \( d \) seen looking toward the load

\[
\bar{Z}(d) = \frac{\bar{V}(d)}{\bar{I}(d)} = \frac{\bar{V}^+ e^{j\beta d} [1 + \bar{\Gamma}(d)]}{(\bar{V}^+/Z_0)e^{j\beta d} [1 - \bar{\Gamma}(d)]}
\]

\[= Z_0 \frac{1 + \bar{\Gamma}(d)}{1 - \bar{\Gamma}(d)}
\]

1. **At the location of a voltage maximum of the standing-wave pattern,** \( 1 + \bar{\Gamma}(d) \) and \( 1 - \bar{\Gamma}(d) \) are purely real and equal to their maximum and minimum magnitudes \( 1 + |\bar{\Gamma}_R| \) and \( 1 - |\bar{\Gamma}_R| \), respectively. Hence, \( \bar{Z}(d) \) is purely real and maximum, say, \( R_{\text{max}} \), equal to \( Z_0[(1 + |\bar{\Gamma}_R|)/(1 - |\bar{\Gamma}_R|)] \), or \( Z_0 \) (SWR).

2. **At the location of a voltage minimum of the standing-wave pattern,** \( 1 + \bar{\Gamma}(d) \) and \( 1 - \bar{\Gamma}(d) \) are purely real and equal to their minimum and maximum magnitudes \( 1 - |\bar{\Gamma}_R| \) and \( 1 + |\bar{\Gamma}_R| \), respectively. Hence, \( \bar{Z}(d) \) is purely real and minimum, say, \( R_{\text{min}} \), equal to \( Z_0[(1 - |\bar{\Gamma}_R|)/(1 + |\bar{\Gamma}_R|)] \), or \( Z_0/(\text{SWR}) \).
7.2 Line Terminated by Arbitrary Load

**Line impedance**

3. Between voltage maxima and minima, \(1 + \Gamma(d)\) and \(1 - \Gamma(d)\) are both complex and out of phase. Hence, \(Z(d)\) is complex, with magnitude lying between \(Z_0\) (SWR) and \(Z_0/(SWR)\).

4. Since \(\Gamma(d + n\lambda/2) = \Gamma(d)e^{\pi j2n\lambda/2} = \Gamma(d)e^{\pi j2n\pi} = \Gamma(d), n = 1, 2, 3, \ldots\), \(\Gamma(d)\) repeats at intervals of \(\lambda/2\), and hence, \(Z(d)\) repeats at intervals of \(\lambda/2\).

5. The product of the line impedances at two values of \(d\) separated by \(\frac{\lambda}{4}\) is given by

\[
\begin{align*}
[\bar{Z}(d)] \begin{bmatrix} Z \left( d \pm \frac{\lambda}{4} \right) \end{bmatrix} &= \begin{bmatrix} Z_0 & 1 + \Gamma(d) & 1 + \Gamma(d) \\ Z_0 & 1 - \Gamma(d) & 1 - \Gamma(d) \end{bmatrix} \begin{bmatrix} 1 + \Gamma(d + \frac{\lambda}{4}) \\ 1 - \Gamma(d + \frac{\lambda}{4}) \end{bmatrix} \\
&= Z_0^2 \begin{bmatrix} 1 + \Gamma(d) \\ 1 - \Gamma(d) \end{bmatrix} \begin{bmatrix} 1 + \Gamma(d) e^{\pi j2\beta \lambda/4} \\ 1 - \Gamma(d) e^{\pi j2\beta \lambda/4} \end{bmatrix} \\
&= Z_0^2 \begin{bmatrix} 1 + \Gamma(d) \\ 1 - \Gamma(d) \end{bmatrix} \begin{bmatrix} 1 + \Gamma(d) e^{\pi j\beta \lambda/4} \\ 1 - \Gamma(d) e^{\pi j\beta \lambda/4} \end{bmatrix} \end{align*}
\]


\[
[\bar{Z}(d)] \begin{bmatrix} Z \left( d \pm \frac{\lambda}{4} \right) \end{bmatrix} = Z_0^2
\]
7.2 Line Terminated by Arbitrary Load

**Input impedance**

\[
\overline{Z}_{in} = \overline{Z}(l) = Z_0 \frac{1 + \overline{\Gamma}(l)}{1 - \overline{\Gamma}(l)}
\]

The input impedance is a useful parameter, because, for a given generator voltage and internal impedance, the power flow down the line can be computed by considering the line voltage and current at any value of \( d \).

Ex: 7.3 (Please refer to textbook for detail).
7.2 Line Terminated by Arbitrary Load

**Normalized impedance and Admittance**

Define:

\[
\tilde{z}(d) = \frac{Z(d)}{Z_0} = \frac{1 + \Gamma(d)}{1 - \Gamma(d)}
\]

\[
\tilde{\Gamma}(d) = \frac{\tilde{z}(d) - 1}{\tilde{z}(d) + 1}
\]

Admittance:

\[
\tilde{Y}(d) = \frac{1}{\tilde{Z}(d)} = \frac{1}{Z_0} \left( \frac{1 - \tilde{\Gamma}(d)}{1 + \tilde{\Gamma}(d)} \right)
\]

\[
\tilde{Y}(d) = Y_0 \frac{1 - \tilde{\Gamma}(d)}{1 + \tilde{\Gamma}(d)}
\]

\[
\tilde{y}(d) = \frac{\tilde{Y}(d)}{Y_0} = \frac{1 - \tilde{\Gamma}(d)}{1 + \tilde{\Gamma}(d)}
\]
7.3 Transmission-Line Matching

When the load impedance is not equal to the characteristic impedance, the input impedance of the line will vary with frequency.

This sensitivity to frequency increases with the electrical length of the line. Why?

To show this, let the length of the line be \( l = n\lambda \). If the frequency is changed by an amount \( \Delta f \), then the change in \( n \) is given by

\[
\Delta n = \Delta \left( \frac{l}{\lambda} \right) = \Delta \left( \frac{lf}{v_p} \right) = \frac{l}{v_p} \Delta f = \frac{n\lambda}{v_p} \Delta f = n \frac{\Delta f}{f}
\]

It is desirable to eliminate standing waves on the line by connecting a matching device near the load such that the line views an effective impedance equal to its own characteristic impedance on the generator side of the matching device.

Does the matching device at the same time absorb any power?

It should be noted that matching, as referred to here, is not related to maximum power transfer since the condition for maximum power transfer is that the line input impedance must be the complex conjugate of the generator internal impedance.
7.3 Transmission-Line Matching

Quarter-Wave Transformer Matching

The quarter-wave transformer, or QWT, matching technique makes use of a section of length $\lambda/4$ of a line of characteristic impedance $Z_q$ different from that of the main line.

We try to decide $d_q$ and $Z_q$.

Where is the possible locations of $d_q$?

$$\overline{Z}_1 \overline{Z}_2 = \overline{Z}_q^2$$

$$\overline{Z}_2 = \frac{\overline{Z}_q^2}{\overline{Z}_1} = \frac{\overline{Z}_q^2}{\overline{Z}_0}$$

Quarter wave transformer

Real number

Do you know now where $d_q$ is $\frac{\lambda}{4}$?
7.3 Transmission-Line Matching

Quarter-Wave Transformer Matching

The line impedance is **purely real** at locations of voltage **maxima** and **minima** of the standing-wave pattern.

Therefore, within the first half-wavelength from the load, there are **two solutions** for $d_q$. 

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7.3 Transmission-Line Matching

Quarter-Wave Transformer Matching

If we choose a voltage minimum for the first solution,

\[ d_q^{(1)} = \frac{\lambda}{4\pi} (\theta + \pi) \]

\[ Z_0 \cdot Z_0 \cdot \frac{1 - |\Gamma_R|}{1 + |\Gamma_R|} = Z_q^2 \]

\[ Z_q^{(1)} = Z_0 \sqrt{\frac{1 - |\Gamma_R|}{1 + |\Gamma_R|}} \]

For the second solution, the value of \( d_q \) corresponds to the location of a voltage maximum that occurs at \( \pm \lambda / 4 \) from the location of the voltage minimum.

\[ d_q^{(2)} = d_q^{(1)} \pm \frac{\lambda}{4} \quad \text{and} \quad Z_q^{(2)} = Z_0 \sqrt{\frac{1 + |\Gamma_R|}{1 - |\Gamma_R|}} \]
7.3 Transmission-Line Matching

Single-stub Matching

In the single-stub matching technique, one stub is used and a match is achieved by varying the location of the stub and the length of the stub.

We shall assume the characteristic impedance of the stub to be the same as that of the line.
7.3 Transmission-Line Matching

**Single-stub Matching**

we observe that to achieve a match, \( \bar{y}_1 \) must be equal to \( 1 + j0 \).

\[
\bar{y}_1 = 1 - jb
\]

\[
\bar{\Gamma}_1 = \frac{1 - \bar{y}_1}{1 + \bar{y}_1} = \frac{jb}{2 - jb}
\]

\[
\bar{\Gamma}_R = \bar{\Gamma}_1 e^{j2\beta d_s} = \frac{jb}{2 - jb} e^{j2\beta d_s} = \frac{|b|}{\sqrt{4+b^2}} e^{j(\pi/2+\tan^{-1}b/2+2\beta d_s+2n\pi)} \quad \text{for } b < 0
\]

\[
|\bar{\Gamma}_R| = \frac{|b|}{\sqrt{4+b^2}} \quad \theta = \pm \frac{\pi}{2} + \tan^{-1} \frac{b}{2} + 2\beta d_s + 2n\pi \quad \text{for } b > 0
\]
7.3 Transmission-Line Matching

Single-stub Matching

\[ b = \pm \frac{2|\Gamma_R|}{\sqrt{1 - |\Gamma_R|^2}} \]

\[ d_s = \frac{\lambda}{4\pi} \left( \theta + \frac{\pi}{2} - \tan^{-1} \frac{b}{2} - 2n\pi \right) \quad \text{for } b > 0 \]

- Two solutions are possible for \( b \)
- The integer value for \( n \) is chosen such that \( 0 \leq d_s < \lambda / 2 \)
7.3 Transmission-Line Matching

Single-stub Matching

To find the solutions for the stub length

\[ \frac{1}{jb} = j \tan \beta l_s \]

Normalized input impedance of a short-circuited line

so

\[ \tan \beta l_s = -\frac{1}{b} \]

\[
\begin{align*}
I_s &= \left\{ \begin{array}{ll}
\frac{\lambda}{2\pi} \tan^{-1} \left( -\frac{1}{b} \right) & \text{for } b > 0 \\
\frac{\lambda}{2\pi} & \text{for } b < 0
\end{array} \right. \\
&= \left\{ \begin{array}{ll}
\frac{\lambda}{2\pi} \left[ \tan^{-1} \left( -\frac{1}{b} \right) + \frac{\lambda}{2} \right] & \text{for } b > 0 \\
\frac{\lambda}{2\pi} & \text{for } b < 0
\end{array} \right.
\end{align*}
\]
7.3 Transmission-Line Matching

Double-stub Matching

In the single-stub matching technique, it is necessary to vary the distance between the stub and the load, as well as the length of the stub, in order to achieve a match for different loads or for different frequencies. This can be inconvenient for some arrangements of lines.

When two stubs are used, it is possible to fix their locations and achieve a match for a wide range of loads by adjusting the lengths of the stubs.

Matching problem then consists of finding the values of \( l_1 \) and \( l_2 \) for a given set of values of \( \bar{Z}_R \) (and hence, of \( \Gamma_R \)), \( d_1 \), and \( d_{12} \).
### 7.3 Transmission-Line Matching

#### Double-stub Matching

\[
\bar{y}_2 = y_2 - jb_2 = 1 - jb_2 \quad \Rightarrow \quad \bar{\Gamma}_2 = \frac{1 - y_2}{1 + y_2} = \frac{jb_2}{2 - jb_2}
\]

\[
\bar{\Gamma}_1 = \bar{\Gamma}_2 e^{j2\beta d_{12}} = \frac{jb_2}{2 - jb_2} e^{j2\beta d_{12}}
\]

\[
\bar{y}_1 = y_1 - jb_1 = \frac{1}{1 - b_2 \sin 2\beta d_{12} + b_2^2 \sin^2 \beta d_{12}} + j \left( \frac{b_2^2 \sin 2\beta d_{12} - 2b_2 \cos 2\beta d_{12}}{2 - 2b_2 \sin 2\beta d_{12} + 2b_2^2 \sin^2 \beta d_{12}} - b_1 \right)
\]
7.3 Transmission-Line Matching

Double-stub Matching

Noting that \( b_1 \) does not appear in the real part expression, we can first compute \( b_2 \) by solving the equation for the real parts. Thus, letting the real part of \( \overline{y}_1 \) as computed from \( \overline{z}_R \) and \( d_1 \) to be \( g' \), we have

\[
\frac{1}{1-b_2 \sin 2\beta d_{12} + b_2^2 \sin^2 \beta d_{12}} = g'
\]

\[
b_2 = \frac{\sin 2\beta d_{12} \pm \sqrt{\sin^2 2\beta d_{12} - 4(1-1/g')\sin^2 \beta d_{12}}}{2\sin^2 \beta d_{12}}
\]

or

\[
b_2 = \frac{\cos \beta d_{12} \pm \sqrt{1/g' - \sin^2 \beta d_{12}}}{\sin \beta d_{12}}
\]
7.3 Transmission-Line Matching

Double-stub Matching

We now see that a solution does not exist for \( b_2 \) if \( g' > \frac{\sqrt{2}}{\sin^2 \beta d_{12}} \), and hence, it is not possible to achieve a match for loads that result in the real part of \( \tilde{Y}_1' \) being greater than \( \frac{1}{\sin^2 \beta d_{12}} \). A simple way to get around this problem is to increase \( d_1 \) by \( \lambda/4 \) (see Problem 7.24).

From the equation for the imaginary parts of \( \tilde{Y}_1' \), the corresponding values of \( b_j \) are then given by

\[
b_1 = \frac{b_2^2 \sin 2\beta d_{12} - 2b_2 \cos 2\beta d_{12}}{2 - 2b_2 \sin 2\beta d_{12} + 2b_2^2 \sin^2 \beta d_{12}} \cdot b'
\]

Finally, the lengths of the two stubs are computed from \( b_1 \) and \( b_2 \)
7.3 Transmission-Line Matching

Bandwidth

For any transmission-line matched system, the match is disturbed as the frequency is varied from that at which the various electrical lengths and distances are equal to the computed values for achieving the match.

For example, in the QWT matched system, the electrical length of the QWT departs from one-quarter wavelength as the frequency is varied from that at which the match is achieved, and the system is no longer matched even if the load does not vary with frequency.
7.3 Transmission-Line Matching

**SWR versus frequency computation**

To discuss a procedure by means of which the SWR versus frequency curve can be computed for all three types of matching techniques discussed, let us consider a transmission-line system having \( n \) discontinuities.

We shall consider a specification of zero for the length of the stub to mean no stub is present instead of a stub of zero length.

---

**TABLE 7.2** Values of Parameters for Using the System of Fig. 7.20 for Three Different Cases

<table>
<thead>
<tr>
<th>System</th>
<th>( n )</th>
<th>( Z_{01} )</th>
<th>( d_1 )</th>
<th>( l_1 )</th>
<th>( Z_{02} )</th>
<th>( d_2 )</th>
<th>( l_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>QWT</td>
<td>2</td>
<td>( Z_0 )</td>
<td>( d_q )</td>
<td>0( ^a )</td>
<td>( Z_q )</td>
<td>1/4</td>
<td>0( ^a )</td>
</tr>
<tr>
<td>Single stub</td>
<td>1</td>
<td>( Z_0 )</td>
<td>( d_s )</td>
<td>( l_s )</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>Double stub</td>
<td>2</td>
<td>( Z_0 )</td>
<td>( d_1 )</td>
<td>( l_1 )</td>
<td>( Z_0 )</td>
<td>( d_{12} )</td>
<td>( l_2 )</td>
</tr>
</tbody>
</table>
7.3 Transmission-Line Matching

SWR versus frequency computation

For a given \( f/f_0 \), the procedure consists of starting at the load and computing in succession the line admittance to the left of the stub at each discontinuity, from a knowledge of the line admittance at the output of the line section to the right of that stub, until the line admittance to the left of the last discontinuity is found and used to compute the required SWR.

To calculate the frequency response, the following formulation is commonly used.

\[
\overline{y}_i = \frac{1 - \overline{\Gamma}_i}{1 + \overline{\Gamma}_i} = \frac{1 - \overline{\Gamma}_o e^{-2\beta l}}{1 + \overline{\Gamma}_o e^{-2\beta l}}
\]

\[
= \frac{1 - [(1 - \overline{y}_0)/(1 + \overline{y}_0)] e^{-2\beta l}}{1 + [(1 - \overline{y}_0)/(1 + \overline{y}_0)] e^{-2\beta l}}
\]

\[
\overline{y}_i = j \sin \beta l + \overline{y}_0 \cos \beta l
\]

\[
\cos \beta l + j \overline{y}_0 \sin \beta l
\]

where \( \overline{\Gamma}_i \) and \( \overline{\Gamma}_o \) are the reflection coefficients at the input and output ends, respectively.
7.4 The Smith Chart: Basic Procedure

The Smith chart is a transformation from the complex $\bar{Z}$-plane (or $\bar{Y}$-plane) to the complex $\bar{\Gamma}$-plane.

We begin with the relationship for the reflection coefficient in terms of the normalized line impedance as given by

\[
\bar{\Gamma}(d) = \frac{\bar{z}(d) - 1}{\bar{z}(d) + 1}
\]

Letting $\bar{z}(d) = r + jx$

\[
\bar{\Gamma}(d) = \frac{r + jx - 1}{r + jx + 1} = \frac{(r-1) + jx}{(r+1) + jx}
\]

\[
|\bar{\Gamma}(d)| = \left|\frac{(r-1)^2 + x^2}{(r+1)^2 + x^2}\right|^{\frac{1}{2}} \leq 1
\]

\[
\bar{\Gamma} = \frac{r + jx - 1}{r + jx + 1} = \frac{r^2 - 1 + x^2}{(r+1)^2 + x^2} + j\frac{2x}{(r+1)^2 + x^2}
\]

\[
\text{Re}(\bar{\Gamma}) = \frac{r^2 - 1 + x^2}{(r+1)^2 + x^2} \quad \text{Im}(\bar{\Gamma}) = \frac{2x}{(r+1)^2 + x^2}
\]
7.4 The Smith Chart: Basic Procedure

**Z is purely real**

\[
\text{Re}(\Gamma) = \frac{r - 1}{r + 1} \quad \text{and} \quad \text{Im}(\Gamma) = 0
\]

**Z is purely imaginary**

\[
|\Gamma| = \left| \frac{x^2 - 1}{x^2 + 1} + j\frac{2x}{x^2 + 1} \right| = 1
\]

\[
\angle \Gamma = \tan^{-1} \left( \frac{2x}{x^2 - 1} \right)
\]
7.4 The Smith Chart: Basic Procedure

$Z$ is complex and its real part is constant

$$
\left[ \text{Re}(\Gamma) - \frac{r}{r+1} \right]^2 + [\text{Im}(\Gamma)]^2
= \left[ \frac{r^2 - 1 + x^2}{(r + 1)^2 + x^2} - \frac{r}{r+1} \right]^2 + \left[ \frac{2x}{(r + 1)^2 + x^2} \right]^2 = \left( \frac{1}{r+1} \right)^2
$$

$Z$ is complex and its imaginary part is constant

$$
[\text{Re}(\Gamma) - 1]^2 + \left[ \text{Im}(\Gamma) - \frac{1}{x} \right]^2
= \left[ \frac{r^2 - 1 + x^2}{(r + 1)^2 + x^2} - 1 \right]^2 + \left[ \frac{2x}{(r + 1)^2 + x^2} - \frac{1}{x} \right]^2 = \left( \frac{1}{x} \right)^2
$$
7.4 The Smith Chart: Basic Procedure

http://www.sss-mag.com/smith.html

Smith chart Collection
7.4 The Smith Chart: Basic Procedure

Example 7.4

A transmission line of characteristic impedance 50 Ω is terminated by a load impedance \( Z_R = (15 - j20) \) Ω. It is desired to find the following quantities by using the Smith chart.

1. Reflection coefficient at the load
2. SWR on the line
3. Distance of the first voltage minimum of the standing-wave pattern from the load
4. Line impedance at \( d = 0.05\lambda \)
5. Line admittance at \( d = 0.05\lambda \)
6. Location nearest to the load at which the real part of the line admittance is equal to the line characteristic admittance

1. \( \bar{\tau}_R = \frac{Z_R}{Z_0} = \frac{15 - j20}{50} = 0.3 - j0.4 \)

2. \( \text{SWR} = \frac{V_{\text{max}}}{V_{\text{min}}} = \frac{(1 + |\Gamma|)}{(1 - |\Gamma|)} \)
   
   at the position of voltage maximum \( B \)

   \( = \frac{R_{\text{max}}}{Z_0} \)

   Normalized input impedance at voltage maximum

   \( = 4 \) in this case

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Distance of the first voltage minimum of the standing-wave pattern from the load
7.4 The Smith Chart: Basic Procedure

example 7.4

4. Line impedance at \( d = 0.05\lambda \)
7.4 The Smith Chart: Basic Procedure

example 7.4

5. Line admittance at \( d = 0.05\lambda \)

\[
\begin{bmatrix}
\bar{Z}(d) \\
\bar{Z}(d + \frac{\lambda}{4})
\end{bmatrix}
\begin{bmatrix}
\bar{Z}(d) \\
\bar{Z}(d + \frac{\lambda}{4})
\end{bmatrix}
= Z_0^2
\]

\[
\begin{bmatrix}
\bar{z}(d) \\
\bar{z}(d + \frac{\lambda}{4})
\end{bmatrix}
\begin{bmatrix}
\bar{z}(d) \\
\bar{z}(d + \frac{\lambda}{4})
\end{bmatrix}
= 1
\]

\[
\bar{y}(d) = \bar{z}(d + \frac{\lambda}{4})
\]
6. Location nearest to the load at which the real part of the line admittance is equal to the line characteristic admittance

4. \(0.325 - 0.185 = 0.14 \lambda\)
7.5 THE SMITH CHART: 2. APPLICATIONS

ex: 7.5 Single-stub matching solution

To achieve a match,
• The stub must be located at a point on the line at which the real part of the normalized line admittance is equal to unity;
• The imaginary part of the line admittance at that point is then canceled by appropriately choosing the length of the stub.
7.5 THE SMITH CHART: 2. APPLICATIONS

ex: 7.5 Single-stub matching solution

(a) Find the normalized load impedance.
7.5 THE SMITH CHART: 2. APPLICATIONS

**ex: 7.5** Single-stub matching solution

\[ y = 1 + j \times 1.16 \]

The location of the stub from the load is

\[ (0.1665 - 0.125)\lambda = 0.0415\lambda. \]
7.5 THE SMITH CHART: 2. APPLICATIONS

ex: 7.5 Single-stub matching solution

The length of the short-circuited stub is $(0.363 - 0.25)\lambda = 0.113\lambda$. Is there a second solution?
7.5 THE SMITH CHART: 2. APPLICATIONS

ex: 7.5 Single-stub matching solution

Second solution
7.5 THE SMITH CHART: 2. APPLICATIONS

ex: 7.6 double-stub matching solution

Auxiliary circle

has to be designed on the auxiliary circle.
7.5 THE SMITH CHART: 2. APPLICATIONS

ex: 7.6 double-stub matching solution

Considering now the numerical values of $\bar{z}_R = (30 - j40)/50 = (0.6 - j0.8)$

$d_1 = 0$, and $d_{12} = 0.375\lambda$,

\[ jb_1 = \bar{y}_1 - \bar{y}_1' = (0.6 - j0.09) - (0.6 + j0.8) = -j0.89 \]

A – B – C – D

\[ jb_2 = -j[\text{Im}(\bar{y}_2')] = j0.53 \]

So,

\[ l_1, \text{ length of first stub} = (0.385 - 0.25)\lambda = 0.135\lambda \]

\[ l_2, \text{ length of second stub} = (0.077 + 0.25)\lambda = 0.327\lambda \]
The second solution:

\[ jb_1 = (0.6 - j1.92) - (0.6 + j0.8) = -j2.72 \]
\[ jb_2 = -j2.5 \]

\[ l_1 = (0.306 - 0.25)\lambda = 0.056\lambda \]
\[ l_2 = (0.31 - 0.25)\lambda = 0.06\lambda \]
7.5 THE SMITH CHART: 2. APPLICATIONS

ex: 7.6 double-stub matching solution

Before proceeding further, we recall from Section 7.3 that in the double stub matching technique, it is not possible to achieve a match for loads that result in the real part of \( \bar{y}_1 \) being greater than \( \frac{1}{\sin^2 \beta d_{12}} \).

In this case, \( d_{12} = 3/8 \lambda \), no solution for the case of

\[
 r > \left( \frac{1}{\sin^2 \beta d_{12}} \right) = 2
\]

This shaded region

Why?

The shaded region is therefore called the forbidden region of \( \bar{y}_1 \) for possible match.

How to solve?
As pointed out in Section 7.3, a solution to the problem is to increase $d_4$ by $\lambda/4$. This effectively rotates the forbidden region by $180^\circ$ about the center of the chart, thereby making possible a match.
7.5 THE SMITH CHART: 2. APPLICATIONS

Transformation of across a discontinuity

\[ y_1 = \frac{\bar{Y}_1}{Y_{01}} = \frac{\bar{Y}_2}{Y_{01}} + \frac{\bar{Y}_d}{Y_{01}} = \frac{Y_{02}}{Y_{01}} \left( \frac{\bar{Y}_2}{Y_{02}} + \frac{\bar{Y}_d}{Y_{02}} \right) \]

\[ 1 - \bar{\Gamma}_1 = a \left( 1 - \frac{\bar{\Gamma}_2}{1 + \bar{\Gamma}_2} + \bar{y}_d \right) \]

\[ \bar{\Gamma}_1 = \frac{(1 + a - a\bar{y}_d)\bar{\Gamma}_2 + (1 - a + a\bar{y}_d)}{(1 - a + a\bar{y}_d)\bar{\Gamma}_2 + (1 + a + a\bar{y}_d)} \]
7.5 THE SMITH CHART: 2. APPLICATIONS

Transformation of across a discontinuity

**Bilinear transformation** between two complex planes, a property of which is that circles in one plane are transformed into circles in the second plane.

\[
\overline{\Gamma_1} = \frac{(1+a-ay_d)\overline{\Gamma_2}+(1-a+ay_d)}{(1-a+ay_d)\overline{\Gamma_2}+(1+a+ay_d)}
\]

Since a circle is defined completely by three points. It is therefore sufficient if we use any three points on the locus of \(y_2\) and find the corresponding three points for \(y_1\).
7.5 THE SMITH CHART: 2. APPLICATIONS

Transformation of across a discontinuity

Ex 7.7

\[ jb = j0.8 \]
\[ y_R = 0.6 + j0.8 \]

We wish to
- find the locus of the normalized admittance \( y_1 \) to the left of the discontinuity as the susceptance slides along the line, and then
- determine the location, nearest to the load, of the susceptance for which the SWR to the left of it is minimized.
7.5 THE SMITH CHART: 2. APPLICATIONS

Transformation of across a discontinuity

A, B, C points on $y_2$ transform to C, D, E, respectively, on $y_1$ based on $y_1 = y_2 + j 0.8$

Where is the minimum SWR point on Locus of $y_1$?

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7.5 THE SMITH CHART: 2. APPLICATIONS
Transformation of across a discontinuity

Point G because of minimum reflection coefficient.

Back to $y_2$ by the bilinear transform
$y_2 = y_1 - j \cdot 0.8$

The slide distance from the load is from point A to H with

$$(0.346 - 0.125) \lambda, \text{ or } 0.221 \lambda.$$
7.6 The Lossy Line

Distributed Equivalent Circuits

The power dissipation in the conductors is taken into account by a resistance in series with the inductor, whereas the power dissipation in the dielectric is taken into account by a conductance in parallel with the capacitor.

\[
V(z + \Delta z, t) - V(z, t) = -R\Delta z I(z, t) - L\Delta z \frac{\partial I(z, t)}{\partial t}
\]

\[
I(z + \Delta z, t) - I(z, t) = -G\Delta z V(z, t) - C\Delta z \frac{\partial V(z, t)}{\partial t}
\]
7.6 The Lossy Line

Transmission Line Equations and Solutions

\[
\begin{align*}
\frac{\partial V(z,t)}{\partial z} &= -RI(z,t) - L \frac{\partial I(z,t)}{\partial t} \\
\frac{\partial I(z,t)}{\partial z} &= -GV(z,t) - C \frac{\partial V(z,t)}{\partial t}
\end{align*}
\]

The corresponding equations in terms of phasor voltage and current

\[
\begin{align*}
\frac{\partial \bar{V}}{\partial z} &= -(R + j\omega L)\bar{I} \\
\frac{\partial \bar{I}}{\partial z} &= -(G + j\omega C)\bar{V} \\
\frac{\partial^2 \bar{V}}{\partial z^2} &= -(R + j\omega L) \frac{\partial \bar{I}}{\partial z} \\
&= (R + j\omega L)(G + j\omega C)\bar{V} \\
\frac{\partial^2 \bar{V}}{\partial z^2} &= \gamma^2 \bar{V}
\end{align*}
\]

\[
\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}
\]
7.6 The Lossy Line

Transmission Line Equations and Solutions

\[ \overline{V}(z) = \overline{A}e^{-\gamma z} + \overline{B}e^{\gamma z} \]

where the superscripts + and − denote (+) and (−) waves, respectively. The quantity \( \beta \), which is the imaginary part of \( \gamma \), is, of course, the phase constant, that is, the rate of change of phase with \( z \) for a fixed time, for either wave. The quantity \( \alpha \), which is the real part of \( \gamma \), is the attenuation constant, denoting that the waves get attenuated by the factor \( e^{\alpha} \) per unit distance as they propagate in their respective directions. Thus, the quantity \( \gamma = \alpha + j\beta \) is the propagation constant associated with the wave.

\[ \overline{I}(z) = -\frac{1}{R + j\omega L} \frac{d\overline{V}}{dz} \]

\[ = -\frac{1}{R + j\omega L} \left[ -\gamma \overline{V}^+ e^{-\gamma z} + \gamma \overline{V}^- e^{\gamma z} \right] \]

\[ \overline{I}(z) = \frac{1}{Z_0} \left[ \overline{V}^+ e^{-\gamma z} + \overline{V}^- e^{\gamma z} \right] \]

where \( Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \) is the characteristic impedance of the line, which is now complex.

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### 7.6 The Lossy Line

**Low Loss Line**

For the special case of the low-loss line:

\[ \omega L \gg R \quad \omega C \gg G \]

\[ \gamma = \sqrt{j\omega L(1 + \frac{R}{j\omega L})j\omega C(1 + \frac{G}{j\omega C})} \]

\[ \approx j\omega \sqrt{LC} \sqrt{1 + \frac{R}{j\omega L} + \frac{G}{j\omega C}} \]

\[ \approx j\omega \sqrt{LC} \left[1 + \frac{1}{2} \left(\frac{R}{j\omega L} + \frac{G}{j\omega C}\right)\right] \]

\[ \approx \frac{1}{2} \left(\frac{C}{L} R + \frac{L}{C} G\right) + j\omega \sqrt{LC} \]

\[ \alpha \approx \frac{1}{2} \left(\frac{C}{L} R + \frac{L}{C} G\right) \]

\[ \beta \approx \omega \sqrt{LC} \]

\[ \nu_p = \frac{\omega}{\beta} \approx \frac{1}{\sqrt{LC}} \]

Thus, for the low-loss line, the expressions for \( \beta \) and \( Z_0 \) are essentially the same as those for a lossless line.
7.6 The Lossy Line

Experimental Determination of $\beta$ and $Z_0$

The experimental technique is based on the measurements of the input impedance of the line for two values of load impedance.

\[
\bar{V}(d) = V^+ e^{j\gamma_d} + V^- e^{-j\gamma_d}
\]

\[
\bar{I}(d) = \frac{1}{Z_0} \left[ V^+ e^{j\gamma_d} + V^- e^{-j\gamma_d} \right]
\]

where

\[
\Gamma(d) = \frac{V^-(d)}{V^+(d)} = \frac{V^- e^{-j\gamma_d}}{V^+ e^{j\gamma_d}}
\]

\[
= \Gamma_R e^{-2\gamma_d} = \Gamma_R e^{-2\alpha d} e^{-j2\beta d}
\]

The line impedance is given by

\[
\bar{Z}(d) = \frac{\bar{V}(d)}{\bar{I}(d)} = Z_0 \frac{1 + \Gamma(d)}{1 - \Gamma(d)}
\]

\[
= Z_0 \frac{1 + \Gamma_R e^{-2\gamma_d}}{1 - \Gamma_R e^{-2\gamma_d}}
\]
7.6 The Lossy Line

Experimental Determination of $\beta$ and $Z_0$

The input impedance of a line of length $l$ terminated by a load impedance $Z_R$ is

$$\bar{Z}_{in} = \bar{Z}(l) = Z_0 \frac{1 + \Gamma_R e^{-2\gamma l}}{1 - \Gamma_R e^{-2\gamma l}}$$

$$= Z_0 \frac{1 + \left[ (Z_R - \bar{Z}_0) / (\bar{Z}_R + \bar{Z}_0) \right] e^{-2\gamma l}}{1 - \left[ (Z_R - \bar{Z}_0) / (\bar{Z}_R + \bar{Z}_0) \right] e^{-2\gamma l}}$$

$$= Z_0 \frac{(Z_R + \bar{Z}_0) + (Z_R - \bar{Z}_0) e^{-2\gamma l}}{(Z_R + \bar{Z}_0) - (Z_R - \bar{Z}_0) e^{-2\gamma l}}$$

$$= Z_0 \frac{Z_R \cosh \gamma l + \bar{Z}_0 \sinh \gamma l}{Z_R \sinh \gamma l + \bar{Z}_0 \cosh \gamma l}$$

$$\bar{Z}_{in} = Z_0 \frac{Z_R + \bar{Z}_0 \tanh \gamma l}{Z_R \tanh \gamma l + \bar{Z}_0}$$

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7.6 The Lossy Line

Experimental Determination of $\beta$ and $Z_0$

Let us now consider two values of $Z_R$: in particular, $Z_R = 0$ and $Z_R = \infty$

$$
\overline{Z}_{in}^s = Z_0 \tanh \overline{\gamma l}
$$

$$
\overline{Z}_{in}^0 = Z_0 \coth \overline{\gamma l}
$$

$$
Z_0 = \sqrt{Z_{in}^s Z_{in}^0}
$$

$$
\tanh \overline{\gamma l} = \frac{Z_{in}^s}{Z_{in}^0}
$$

$\alpha l = 0.3466$

$\alpha = 0.3466 / l$

$\beta l = n\pi - \pi / 4$

However, if the approximate value of $\beta$ is known, then the correct value of $n$, and hence of $\beta$ can be determined.

Ex:

$$
\overline{Z}_{in}^s = (30 - j40)\Omega
$$

$$
\overline{Z}_{in}^0 = (30 + j40)\Omega
$$

$$
\overline{Z}_0 = \sqrt{(30 - j40)(30 + j40)} = 50\Omega
$$

$$
\tanh \overline{\gamma l} = \frac{30 - j40}{30 + j40} = \sqrt{50} \angle -53.13^\circ
$$

$$
= 1 \angle -53.13^\circ = 0.6 - j0.8
$$

$$
\overline{\gamma l} = \tanh^{-1}(0.6 - j0.8)
$$

$$
\tanh^{-1} x = \frac{1}{2} \ln \frac{1 + x}{1 - x}
$$

$$
\overline{\gamma l} = \frac{1}{2} \ln \frac{1.6 - j0.8}{0.4 + j0.8} = \frac{1}{2} \ln \frac{1.789}{0.894} \angle 63.435
$$

$$
= \frac{1}{2} \ln 2 \angle 90^\circ = \frac{1}{2} \ln \left[ 2 e^{j(2n\pi - \pi / 2)} \right]
$$

$$
= \frac{1}{2} \ln 2 + j(2n\pi - \pi / 2) = 0.3466 + j(n\pi - \pi / 4)
$$
7.6 The Lossy Line

Experimental Determination of $\beta$ and $Z_0$

In practice, since a perfect open-circuited termination can often be difficult to achieve, it may be desirable to consider the second value of $Z_R$ to be arbitrary instead of being equal to $\infty$.

\[
\overline{Z_{in}} = \overline{Z_0}^2 \frac{\overline{Z_R} + \overline{Z_{in}}}{\overline{Z_R} \overline{Z_{in}} + \overline{Z_0}^2}
\]

\[
\overline{Z_0} = \sqrt{\frac{\overline{Z_R} \overline{Z_{in}} \overline{Z_{in}}}{\overline{Z_R} + \overline{Z_{in}} - \overline{Z_{in}}}}
\]

\[
\overline{Z_{in}} = \overline{Z_0} \tanh \overline{\gamma l}
\]

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7.6 The Lossy Line

Power Flow

Power flow down the line.

\[
\langle P \rangle = \frac{1}{2} \text{Re} \left[ \overline{V}(d) \overline{I}'(d) \right] \\
= \frac{1}{2} \text{Re} \left\{ \overline{V}' e^{\overline{\eta}d} \left[ 1 + \overline{\Gamma}(d) \right] \frac{(\overline{V} +)}{Z_0} e^{\overline{\eta}d} \left[ 1 - \overline{\Gamma}^*(d) \right] \right\} \\
= \frac{1}{2} \text{Re} \left\{ \frac{\overline{V}^2}{Z_0} e^{2\overline{\eta}d} \left[ 1 - |\overline{\Gamma}(d)|^2 + \overline{\Gamma}(d) - \overline{\Gamma}^*(d) \right] \right\} \\
= \frac{1}{2} \text{Re} \left\{ \overline{V}^2 \overline{Y}_0 e^{2\overline{\eta}d} \left[ 1 - |\overline{\Gamma}(d)|^2 + j2 \text{Im} \overline{\Gamma}(d) \right] \right\} \\

\langle P \rangle = \frac{1}{2} \overline{V}^2 e^{2\overline{\eta}d} \left\{ G_0 \left[ 1 - |\overline{\Gamma}(d)|^2 \right] + 2B_0 \text{Im} \overline{\Gamma}(d) \right\} \\

\text{where} \quad \overline{Y}_0 = \frac{1}{Z_0} = G_0 + jB_0
\]
7.6 The Lossy Line

Power Flow (example)

Please compute the time-average power delivered to the input of the line, the time-average power delivered to the load, and the time-average power dissipated in the line.

\[ Z_g = (30 - j40) \, \Omega \]

\[ V_g = 100 \angle 0^\circ \, V \]

\[ Z_0 = 50 \, \Omega \]

\[ \alpha = 10^{-2} \, \text{Np/}\lambda \]

\[ Z_R = 150 \, \Omega \]

\[ \Gamma_R = \frac{Z_R - Z_0}{Z_R + Z_0} \frac{150 - 50}{150 + 50} = 0.5 \]

\[ \Gamma(l) = \Gamma_R e^{-2\gamma l} = \Gamma_R e^{-2\alpha l} e^{-j2\beta l} \]

\[ = 0.5 e^{-0.204} e^{-j40.8\pi} \]

\[ = 0.4077 \angle -144^\circ \]
7.6 The Lossy Line

Power Flow (example)

The input impedance of the line is given by

\[ \bar{Z}_{in} \bar{Z}(l) = \frac{Z_0 1 + \bar{\Gamma}(l)}{1 - \bar{\Gamma}(l)} \]

\[ = 50 \frac{1 + 0.4077 \angle -144^\circ}{1 - 0.4077 \angle -144^\circ} = 50 \frac{1 + (0.33 - j0.24)}{1 - (0.33 - j0.24)} \]

\[ = 50 \frac{0.7117 \angle -19.708^\circ}{1.3515 \angle 10.299^\circ} = 26.33 \angle - 29.937^\circ \]

\[ = (22.817 - j13.140) \Omega \]

The current drawn from the voltage generator can be obtained as

\[ \bar{I}(l) = \frac{\bar{V}_g}{\bar{Z}_g + \bar{Z}_{in}} = \frac{100 \angle 0^\circ}{(30 - j40) + (22.817 - j13.140)} \]

\[ = \frac{52.817 - 53.140}{74.923 \angle - 45.175^\circ} \]

\[ = 1.3347 \angle 45.175^\circ \]
7.6 The Lossy Line

Power Flow (example)

The voltage at the input end of the line is given by

\[ \overline{V}(l) = Z_{in} \overline{I}(l) = 26.33 \angle -29.937^\circ \times 1.3347 \angle 45.175^\circ \]

\[ = 35.143 \angle 15.238^\circ \]

The time-average power flow at the input end of the line is given by

\[ \langle P(l) \rangle = \frac{1}{2} \text{Re}\left[ \overline{V}(l) \overline{I}^*(l) \right] \]

\[ = \frac{1}{2} \text{Re}\left[ 35.143 \angle 15.238^\circ \times 1.3347 \angle -45.175^\circ \right] \]

\[ = \frac{1}{2} \times 35.143 \times 1.3347 \times \cos 29.937^\circ \]

\[ = 20.32W \]
7.6 The Lossy Line

Power Flow (example)

Noting that \(B_0 = 0\), we then obtain the value of \(|V^+|\) by applying (7.82) to \(d = l\). Thus,

\[
|V^+| = \sqrt{\frac{2\langle P(l)\rangle e^{-2al}}{G_0 \left[1 - \overline{\Gamma(l)}^2\right]}}
\]

\[
= \sqrt{\frac{2 \times 20.32 \times e^{-0.204}}{0.02(1 - 0.4077^2)}} = 44.58 V
\]

The time-average power delivered to the load is then given by

\[
\langle P(0) \rangle = \frac{1}{2} |V^+|^2 G_0 \left(1 - \overline{\Gamma_R}^2\right)
\]

\[
= \frac{1}{2} \times 44.58^2 \times 0.02(1 - 0.25) = 14.91 W
\]

Finally, the time-average power dissipated in the line is

\[
\langle P_d \rangle = \langle P(l) \rangle - \langle P(0) \rangle = 20.32 - 14.91 = 5.41 W
\]
7.7 PULSES ON LOSSY LINES

Since $\alpha$ and $\nu_p$ are in general functions of frequency, the different frequency components of an arbitrarily time-varying signal undergo different amounts of attenuation and different amounts of phase shift. Hence, as the signal propagates down the line, it gets distorted.

For the general case, the analysis can be performed by employing Fourier techniques.

There are, however, two special cases of importance that permit solution without the use of Fourier techniques: distortionless transmission and diffusion.
7.7 PULSES ON LOSSY LINES

Distortionless Line

As the signal propagates down the line, its shape versus time remains the same, but it diminishes in magnitude. The situation arises for the condition:

\[
\frac{R}{L} = \frac{G}{C}
\]

\[
\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}
\]

\[
= \sqrt{R(1 + j \frac{\omega L}{R})G(1 + j \frac{\omega C}{G})}
\]

\[
= \sqrt{RG} \left(1 + j \frac{\omega L}{R}\right)
\]

\[
= \sqrt{RG} + j\omega \sqrt{\frac{G}{R}}
\]

\[
= \sqrt{RG} + j\omega \sqrt{\frac{C}{L}}
\]

\[
= \sqrt{RG} + j\omega \sqrt{LC}
\]
### 7.7 PULSES ON LOSSY LINES

**Distortionless Line**

\[
\alpha = \sqrt{RG} \\
\beta = \omega \sqrt{LC} \\
\nu_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}
\]

\[
Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{R(1 + j\omega L/R)}{G(1 + j\omega C/G)}}
\]

\[
= \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}
\]

\(\alpha\) and \(\nu_p\) are both independent of frequency, provided, of course, that \(R, L, C, G\) are independent of frequency.
7.7 PULSES ON LOSSY LINES

Diffusion

This case pertains to the historically important noninductive, leakage-free cable first investigated by Lord Kelvin in 1855, but is also relevant to modern lines with large skin-effect losses and to many other physical phenomena, such as heat flow.

The noninductive, leakage-free cable is characterized by $L=G=0$ so that the distributed equivalent circuit consists simply of series resistors and shunt capacitors,

$$\frac{\partial V(z,t)}{\partial z} = -RI(z,t)$$
$$\frac{\partial I(z,t)}{\partial z} = -C \frac{\partial^2 V(z,t)}{\partial t}$$

**diffusion equation**

$$\frac{\partial^2 \tau}{\partial z^2} = \frac{1}{D} \frac{\partial \tau}{\partial t}$$
### 7.7 PULSES ON LOSSY LINES

**Diffusion**

\[
\frac{\partial^2 \tau}{\partial z^2} = \frac{1}{D} \frac{\partial \tau}{\partial t}
\]

**Solution:**

\[
\tau = A \int_0^{(\sqrt{1/4Dt})} e^{-Y^2} dY + B
\]

where \( A \) and \( B \) are arbitrary constants to be evaluated from boundary conditions, and \( Y \) is a dummy variable.

**Solution:**

\[
V(z, t) = A \int_0^{(\sqrt{RC/4t})} e^{-Y^2} dY + B
\]

What are \( A \) and \( B \)?
Let us now consider an initially quiescent cable extending from $z = 0$ to $z = \infty$ and excited at $z = 0$ by a constant voltage source of value $V_0$ connected at $t = 0$,

\[ V(0, 0+) = V_0 \]
\[ V(\infty, t) = 0 \quad \text{for} \quad t > 0 \]

\[
A\int_0^0 e^{-y^2} \, dy + B = V_0 \\
A\int_0^\infty e^{-y^2} \, dy + V_0 = 0 \rightarrow A\frac{\sqrt{\pi}}{2} + V_0 = 0 \rightarrow A = -2V_0 \sqrt{\pi}
\]
7.7 PULSES ON LOSSY LINES

Diffusion

Thus the particular solution

\[
V(z, t) = -\frac{2V_0}{\sqrt{\pi}} \int_0^{(\sqrt{RC/4t})z} e^{-y^2} \, dy + V_0
\]

\[
= V_0 \left[ 1 - \frac{2}{\sqrt{\pi}} \int_0^{(\sqrt{RC/4t})z} e^{-y^2} \, dy \right]
\]

\[
= V_0 \left[ 1 - \text{erf} \left( \sqrt{RC/4t}z \right) \right]
\]

\[
= V_0 \text{erfc} \left( \sqrt{RC/4t}z \right)
\]

error function (erf)

http://mathworld.wolfram.com/Erf.html

erfc is the complementary error function.

the corresponding solution for \( I(z, t) \)

\[
I(z, t) = -\frac{1}{R} \frac{\partial V(z, t)}{\partial t}
\]

\[
= V_0 \sqrt{\frac{C}{\pi R t}} e^{-(RC/4t)z^2}
\]

\[
\frac{d}{dz} \text{erf} (z) = \frac{2}{\sqrt{\pi}} e^{-z^2}
\]

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How to solve the diffusion equation?

Laplace transform

\[
\frac{\partial^2 V}{\partial z^2} = RC \frac{\partial V}{\partial t}
\]

\[
\frac{\partial^2 \tilde{V}(z, s)}{\partial z^2} = R C s \tilde{V}(z, s)
\]

\[
\tilde{V}(z, s) = A e^{-\sqrt{s} RC z} + B e^{\sqrt{s} RC z}
\]

Step function excitation at \( z = 0 \)

\[
\tilde{V}(0, s) = \frac{V_0}{s}
\]

B. C.

V = 0 at \( z = \infty \)

\[
\tilde{V}(\infty, s) = 0
\]

It is known the Laplace transform of erfc is (proof is complicated)

\[
erfc\left(\frac{a}{2 \sqrt{t}}\right) \quad \longleftrightarrow \quad \frac{e^{-a \sqrt{s}}}{s}
\]

\[
V(z, t) = V_0 erfc\left(\sqrt{RC / 4t} z\right)
\]
7.7 PULSES ON LOSSY LINES

Diffusion

Error function

Complementary Error function

Erf(z) is the "error function" encountered in integrating the normal distribution (which is a normalized form of the Gaussian function).

\[
\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt
\]

\[
\text{erf}(z) + \text{erfc}(z) = 1
\]
7.7 PULSES ON LOSSY LINES

Diffusion

Since the argument involves both \( z \) and \( t \) in the manner \( \sqrt{\frac{RC}{4t}} \), this sketch represents the shape of the line voltage variation with \( z \) for any fixed value of \( t \).

\[
\frac{V(z,t)}{V_0} = \text{erfc}\left(\sqrt{\frac{RC}{4t}} z\right)
\]

Thus, we note that, immediately after closure of the switch, there is voltage everywhere on the line. This corresponds to the phenomenon of diffusion—as distinguished from propagation, which is characterized by a well-defined velocity.
7.7 PULSES ON LOSSY LINES

Diffusion

As $t$ increases, a given point on the sketch corresponds to larger and larger values of $z$, indicating that as time progresses, the line voltage at all values of $z$ increases.

For example, since the values of $z$ are doubled as $t$ is quadrupled, the voltage at a given distance from the source and at a particular time after closure of the switch is the same as the voltage at half that distance and at one-fourth of that time.
7.7 PULSES ON LOSSY LINES

Diffusion

We can also obtain the time-variations of the line voltage for fixed values of $z$, by noting that for a fixed $z$, the numbers on the abscissa can be converted to values of time.
7.7 PULSES ON LOSSY LINES

Diffusion

Finally, let us consider the voltage source in the system of Fig. 7.40 to be a rectangular pulse of amplitude \( V_0 \) and duration \( t_0 \). Then according to superposition, the rectangular pulse is equivalent to the sum of two constant voltages, one of value \( V_0 \) connected at \( t = 0 \) and the second of value \(-V_0\) connected at \( t = t_0 \). Applying the result of Fig. 7.39(b) to each constant voltage source and using superposition, we can find the response to the rectangular pulse.

Note:
As the value of \( z \) is increased, the attainment of the maximum of the pulse is delayed and the value of the maximum is reduced.